CALT Round 3 Individual

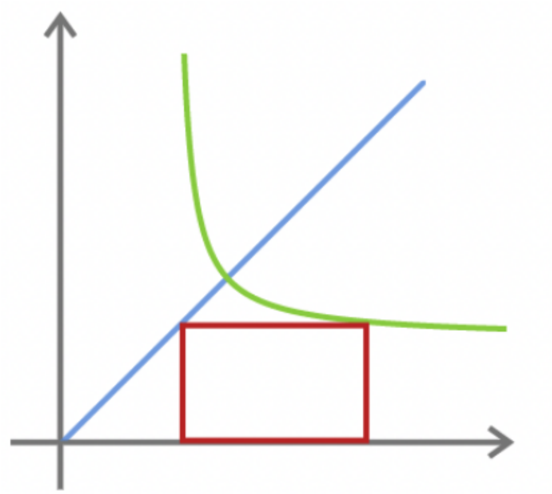
Rules

1. You have 75 minutes to complete this test.
2. You are permitted the use of scratch paper, rulers, protractors, and compasses. No online resources or calculators are allowed.
3. All answers will be integers.
4. Discussion of this test is not permitted.
5. Figures are not necessarily drawn to scale.

May 2021

1. There are four bookshelves in a small library which are labelled #1, #2, #3, and #4 and they hold $120, 135, 142$, and 167 books respectively. A day later, a books have been removed from each shelf. Another day later, $0, b, c$, and d books have been removed from bookshelves #1, #2, #3, and #4 respectively. At that time, all 4 shelves have the same number of books remaining. If b, c, and d are positive integers such that $a = b + c + d$, how many books are remaining on Shelf #1?

2. Suppose rectangle $ABCD$ is located in the first quadrant, so that point A is on the straight line $y = x$, point B is on the curve $y = \frac{1}{x}$ and points C and D are on the x-axis as shown below. If $AB = 2BC$, find the area of rectangle $ABCD$. If your answer can be expressed as $\frac{p}{q}$, where p and q are coprime, find $10p + q$.



3. ATV and Elmo are in a meme competition, and their memes are rated out of 15. Elmo's rating is a random positive integer from 1 to 15 inclusive, while ATV gets the ratings $1, 2, \ldots, 15$ with ratio $1 : 2 : \cdots : 14 : 15$ respectively. If the probability that ATV wins the meme competition can be expressed as $\frac{a}{b}$, where $\gcd(a, b) = 1$, find $a + b$. A tie is not a win.

4. For a real value x, $2^{x^3} = 3^{x^4}$, compute the sum of the possible values of $6^{2^{3^x}}$.

5. Let $f(x) = x^3 + 9x^2 + bx + c$. If the values $9, b, c$ are in an arithmetic sequence and the roots of $f(x)$ are also in an arithmetic sequence, find $b + c$.

6. Let $o_k(x)$ be the area of the regular polygon with k sides and circumradius x. Let $p_k(x)$ be the area of the regular polygon with k sides and side length x. If $\frac{o_{12}(1)}{p_{12}(1)}$ can be expressed as $a - \sqrt{b}$, find $10a + b$.

7. Let $f_k(n)$ denote the number of solutions a to $a^k \equiv 1 \mod n$ where $a < n$ and a is a nonnegative integer. Compute
$$\sum_{p<100, p \text{ is a prime}} f_7(p).$$

8. Let S_m be the area of the triangular region that is enclosed by the straight lines

$$\begin{aligned} l_1 &: y = mx + 2(m-1) \\ l_2 &: y = (m+1)x + 2m \\ l_3 &: y = 0 \end{aligned}$$

for all $m = 1, 2, 3, \ldots$. Find the value of $S_1 + S_2 + \cdots + S_{2020} + S_{2021}$. If your answer can be expressed as $\frac{p}{q}$, where $\gcd(p, q) = 1$, find $p + q$.

9. Dan, Emily and Fiona are painters. Dan can paint a wall in d hours, Emily can paint the same wall in e hours and Fiona can paint the same wall in $\frac{2021}{2020}$ hours. Since Dan and Emily are jealous of Fiona, they decide to work together on a wall and finish it before or at the same time at which Fiona can complete hers, so they can get rid of their jealousy. Given that d and e are positive integers lesser than 50, find the number of ordered pairs (d, e) that make Dan and Emily NOT get rid of their jealousy.

10. As shown in the diagram below, O is the center of a regular hexagon $ABCDEF$ where $OM \perp ED$ at M and $ON = NM$. If the area of triangle FAN is 10, find the area of hexagon $ABCDEF$.