CALT Round 2

Rules

1. You have 60 minutes to complete this test.
2. You are permitted the use of scratch paper, rulers, protractors, and compasses. No online resources or calculators are allowed.
3. All answers will be integers.
4. Discussion of this test is not permitted.
5. Figures are not necessarily drawn to scale.

February 2021

1. Define a recursion such that $a_1 = 1$ and for $n \geq 2$, $a_n = a_{n-1}/3$. Let r be a constant. We define $A(n)$ as the sum of an infinite geometric series with first term a_n and common ratio r. If
$$\sum_{n=1}^{\infty} A(n) = 50,$$
then r can be expressed as $\frac{p}{s}$, where $\gcd(p, s) = 1$, find $p + s$.

2. A regular hexagon and all of its diagonals are drawn. Let the ratio of the smallest region's area to the largest region's area be m, where m can be expressed as $\frac{a}{b}$ with $\gcd(a, b) = 1$. Find $a + b$.

3. Define $s(n)$ as the sum of the digits of n. Let $f(n)$ be the least nonnegative integer value of m in $s^m(n)$, such that the value of $s^m(n)$ is a single digit. For example, $f(47)$ would be 2, as $s^2(47) = 2$ but $s^1(47)$ is 11, which is not a single digit and $f(3) = 0$. Find $f(1) + f(2) + f(3) + \cdots + f(100)$.

4. Triangle ABC has side lengths $AB = 13$, $AC = 14$, and $BC = 15$. A segment PQ is drawn parallel to BC and tangent to the incircle of $\triangle ABC$ with P on AB and Q on AC. If the distance from P to AC can be expressed as $\frac{a}{b}$, where $\gcd(a, b) = 1$, find $a + b$.

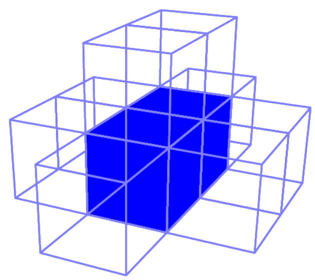
5. If all the ordered pairs (x, y), with x and y positive integers, such that $\operatorname{lcm}(x, y) + \gcd(x, y) = x + y + 26$, can be written as $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, find $x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n$.

6. A random permutation of the word the word "caltop" is chosen. The probability that there is at least one string of

4 consecutive letters that is alphabetically ordered can be expressed as $\frac{a}{720}$. Find a.

7. In trapezoid $ABCD$, with BC parallel to AD and $AB = 4$, $BC = 5$, $CD = 12$ and $DA = \frac{11}{2}$ units, the angle bisectors of A and B meet at point X and the angle bisectors of C and D meet at point Y. If the distance from X to Y can be expressed as $\frac{a}{b}$, where $\gcd(a,b) = 1$, find $a+b$.

8. Consider a 2×1 bed. Define $f(n)$ as the amount of 1×1 blocks you need to cover the bed such that you need to break a minimum of n blocks to reach the bed, such that you can't attack from underneath. For example, $f(1) = 8$ and a diagram of this is shown. Find $f(100) \mod 1000$.



9. Tony lists all the factors of 105 in order. Bryan comes along and randomly selects one of the numbers Tony wrote down, then writes down all the numbers from 1 to that number. For example, if Bryan selects the number 15, he would list the numbers from 1 to 15 in order. Mike then chooses an integer x uniformly at random from the ordered list Bryan wrote and then, Carl selects a number y uniformly at random from 1 to that number x Mike chose. If the expected value of y can be expressed as $\frac{a}{b}$, where $\gcd(a,b) = 1$, find $a+b$.

10. Find the remainder when $\binom{2000}{600} - \binom{1000}{200}$ is divided by 16.