The Second Round

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and the CALT committee

Thanks to

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for testsolving

Rules

1. You have fifteen minutes to complete this test.
2. You are permitted the use of scratch paper, rulers, protractors, and compasses. No online resources or calculators are allowed.
3. All answers will be integers.
4. Discussion of this test is not permitted.
5. Figures are not necessarily drawn to scale.
6. SCORING: The first 4 questions will be worth 5 points each, while the proof problem will be graded out of 10.

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1. Let $(a_n)_{n\geq 1}$ be a sequence of positive integers such that $a_1 = 6$, $a_2 = 66$, and

$$a_n = a_{n-1} + 12a_{n-2}$$

for all $n \geq 3$. For how many integers $1 \leq n \leq 100$ does the units digit of a_n equal 6?

2. Tim has an important test, and while he is very smart, he has trouble under pressure. The test consists of 3 balanced questions. At the beginning of the test, the probability that he gets a question correct is $\frac{2}{3}$; however, every time he gets a question incorrect, this probability is halved. (For exaample, if Tim gets the first and third questions correct but the second question incorrect, the probabilities of solving each question are $\frac{2}{3}$, $\frac{2}{3}$, and $\frac{1}{3}$, respectively.) The expected value of the number of questions he gets correct can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

3. Let $ABCD$ be a square with side length 5. Point P is chosen outside of the square such that $PA = 4$ and $PB = 3$, and Q is the intersection point of lines AP and BD. The area of $\triangle BPQ$ can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

4. Suppose u and v are complex numbers satisfying the system of equations

$$(u-1)(v-1) = 9 \qquad \text{and} \qquad u^3 - u^2 = v^3 - v^2$$

Find the sum of all possible values of $|u|^2 + |v|^2$.

5. For each pair (m, n) of non negative integers with $m \leq n$, let $f(m, n)$ denote the remainder when $1 + 7 + \ldots + 7^n$ is divided by $1 + 7 + \ldots + 7^m$. Find, with proof, the maximum possible value of $f(m, n)$ over all pairs (m, n) with $1 \leq m \leq n \leq 100$.