The First Round

Written by bobthefam

and the CALT committee

Thanks to

djmathman, kootrapali, IMadeYouReadThis, lrjr24, and nikenissan

for testsolving

Rules

1. You have thirty-five minutes to complete this test.
2. You are permitted the use of scratch paper, rulers, protractors, and compasses. No online resources or calculators are allowed.
3. All answers will be integers.
4. Discussion of this test is not permitted.
5. Figures are not necessarily drawn to scale.
6. SCORING: Each question will be worth 2 points on this test.

October 2020

1. Let n be a positive integer. A banana slug is walking across a tightrope. Every $\frac{1}{n}$th of the way across, it gains 30 grams in weight by eating a cracker. The tightrope can only hold 1000 grams of weight. If it's starting weight is 690 grams, what is the largest possible value of n such that the banana slug doesn't break the tightrope?

2. Seven marbles, each of which is either blue, red, or yellow, are in a box. It is known that there is a different amount of marbles for each color and there is at least 1 marble of each color. Steve calculates the probability that the first 3 marbles that he chooses are red. If the difference in the least and greatest probability that Steve can calculate is $\frac{a}{b}$, where a and b are relatively prime, what is $a+b$?

3. Bella and Stella's teacher asks the pair to calculate the value of 314_7 in base 5. Bella calculates the correct answer but Stella accidentally thinks that 314 is in base 5 so she converts 314_5 to base 7. What is the absolute difference of the sum of the digits of the 2 answers?

4. For each positive integer n, let $f(n)$ equal the number of positive integer divisors of n^n, and let $g(n)$ equal the number of perfect squares less than n. Find
$$g(f(20)) - f(g(20)).$$

5. Drew creates a 4-sided tetrahedron dice by randomly choosing numbers from $2-6$. He then takes the 3 numbers that are not on the bottom and creates a 3-digit number using these digits. Then he takes the GCD of all the 6 numbers he can make using those 3 digits. What is the value of the highest GCD he can get?

6. One worker can complete a job in 2 hours, the next can complete in 4 hours, and so on for powers of 2. We hire the first n fastest workers. What is the smallest value of n, such that they complete the job in under 64 minutes?

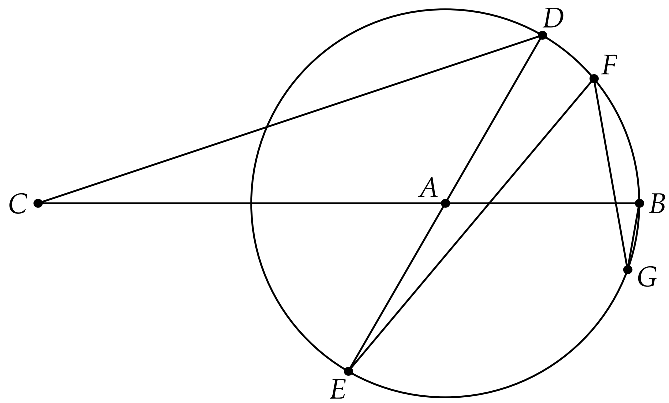
7. Ben draws a quadrilateral with coordinates of $(-1,2),(2,-2),(7,10)$, and $(3,13)$. If the sum of the area and perimeter of this quadrilateral can be expressed as $a+b\sqrt{c}$ where c is not divisible by a square of a prime, what is the value of $a+b+c$?

8. If the numerical area and perimeter of a regular octagon are equal, and the area of the octagon can be expressed as $a\sqrt{b}-c$, find $a+b+c$, such that b is not divisble by the square of any prime.

9. John's family is taking a trip to the theater. There are 6 people going: John, his parents, his grandparents, and his friend. John wants to sit next to his friend. His father wants to spend as much time as he can with John's grandparents so he sits in between them. John's mom has no restrictions and sits where the spot is open. In how many ways can they sit?

10. A frustum is created by slicing a cone parallel to its base. If the frustum has a height of 6, and the radii are 10 and 14, then the volume of the frustrum can be expressed as $a\pi$. Find a.

11. Points B, D, E, F, and G lie on a circle with center A as shown below, and point C lies on line AB outside the circle. Suppose that
$$\angle FGB = 39°, \quad \angle DEF = 11°, \quad \text{and} \quad \angle DCA = 32°.$$
What is the measure of $\angle EDC$?



12. Let a, b, c, d, and e be real numbers satisfying the system of equations

$$\begin{aligned} a+b+c+d+e &= 7, \\ ab+ac+ad+ae+bc+bd+be+cd+ce+de &= -4, \\ abc+abd+abe+acd+ace+ade+bcd+bce+bde+cde &= -28, \\ abcd+abce+abde+acde+bcde &= 1, \\ abcde &= 7. \end{aligned}$$

What is the value of the largest of the 5 variables?

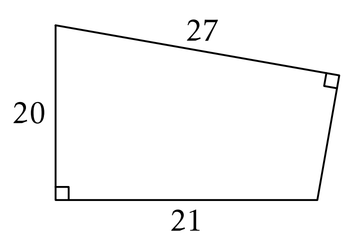
13. Bob is a weatherman for "The World Daily". He is reporting about the temperature in Orlando, Florida, as many tourists are flocking over to Orlando. He states that there is a 60% of chance of rain on Friday, and states if it does rain on Friday, there is a 70% chance of rain on Saturday. If there is no rain on Friday, then there would be a 50% chance of rain on Saturday. If it rains on Saturday, then there is a 80% chance of rain on Sunday. If it does not rain on Saturday, there is a 60% chance of rain on Sunday. If the probability that it rains exactly 2 days is expressed as $\frac{a}{b}$, where a and b are relatively prime, find $a+b$.

14. For each positive integer n, let z_n be the unit digits of the sum of the first n odd positive integers. Find the remainder when

$$\sum_{n=1}^{2020} z_n$$

is divided by 100.

15. The figure below shows a quadrilateral with two right angles. The distance between these two right-angled vertices can be written in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, and d are relatively prime positive integers and c is not divisible by the square of any prime. What is $a+b+c+d$?



16. Suppose a and b are real numbers such that $14a + 2b = 36$. What is the remainder when

$$x^4 - 3x^3 + 2x^2 + ax + b$$

is divided by $x - 7$?

17. Square $ABCD$ has side length 4 units. Let M be the point that is $\frac{1}{3}$ of the way from A to D and let N be the midpoint of the line $\overline{BM}$. If the area of quadrilateral $AMCN$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime, find $a + b$.

18. Jack has 3 unfair coins and each land on tails with a probability of 60%. If it lands on heads, he rolls a fair die but if it lands on tails, he moves on to the next coin. He scores his game in the sense that each die roll is worth the number that is showing up. The coin flip is worth 1 point for each head and 2 points for each tail. What is the expected value of the number of points he gets?

19. A function $f(x)$ is made such that the value of $f(x)$ is x plus the number formed by reversing the digits of x. For example, $f(15) = 15 + 51 = 66$ and $f(20) = 20 + 02 = 22$. Let

$$S = f(1) + f(2) + f(3) + \cdots + f(98) + f(99).$$

What is the sum of the digits of S?

20. Bob has 2 sets of all the integers from 1 through 14, inclusive. He first picks a number n out of the first set and then chooses a number $m \neq n$ uniformly at random from the second set subject to the following conditions:

- If n is greater than 7, then m is less than n;
- If n is less than 7, then m is greater than n;
- If n equals 7, then m is chosen randomly.

(For example, if Bob chose 12 on his first pick, he has to choose a number less than 12 on his second pick.) If the expected value of the sum of the two numbers he picks can be expressed as $\frac{a}{b}$, where a and b are relatively prime, find $a + b$.