Intro to Summations

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May 2021

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§1 Prerequisites

Definition 1.1. An **arithmetic sequence** is a sequence where the difference between any two consecutive terms is the same constant. This constant is called the common difference. An example is

$$1, 4, 7, 10, 13$$

as the common difference is 3 between any two consecutive terms.

Theorem 1.1 (Sum of an Arithmetic Sequence)

The sum of an arithmetic series with first term a_1, last term a_2, common difference d, and n terms is

$$\frac{(a_1 + a_n) \cdot n}{2}.$$

Proof. Lets write out the terms of the arithmetic series in terms of a_1, a_n, n, and d. It is

$$a_1, a_1 + d, a_1 + 2d, \ldots, a_1 + d(n-1).$$

Now, lets add up the first and last terms, the second and second last, the third and third last, and so on. Now, we notice that all these sums add up to the same thing, $2a_1 + dn - d$. How many of these "sets" do we have? We have $\frac{n}{2}$. (Note that if n is odd, the middle term has half the value which accounts for the extra half in $\frac{n}{2}$.) The total sum is $\frac{n}{2} \cdot (2a_1 + dn - d)$, and since $a_n = a + dn - d$, our final value is

$$\boxed{\frac{(a_1 + a_n) \cdot n}{2}}.$$

□

Its important to know a special case of this.

Theorem 1.2 (Sum of the First N Natural Numbers)

The sum of the first n numbers is

$$\frac{n(n+1)}{2}.$$

Proof. This is just an arithmetic sequence, with first term 1, last term n, n terms, and 1 as the common difference. Plugging it in the arithmetic sequence formula, we have

$$\boxed{\frac{n(n+1)}{2}}$$

as desired. □

Now lets explore geometric series.

Definition 1.2. An **geometric series** is a series in which the ratio of any two consecutive terms is the same constant. This constant is called the **common ratio**. An example would be

$$7, 14, 28, 56, 112$$

as it has common ratio 2.

Theorem 1.3 (Sum of Finite Geometric Series)
A geometric series with common ratio r, first term a, and n terms is

$$a\left(\frac{1-r^n}{1-r}\right).$$

Proof. Again, lets write out the sequence. We have

$$a + ar + ar^2 + \cdots + ar^{n-1}.$$

First, lets factor out the a. Now, to derive it, lets take a look at a special polynomial. If we try to multiply the two terms $(1-x)$ and $(1+x+x^2+\cdots+x^n)$ by writing out the first few terms, we notice they cancel. Simplifying, this reduces to $1-x^{n+1}$. The original series was

$$a(1+r+r^2+\cdots+r^{n-1})$$

which is very similar to the polynomial we just observed. We realize that

$$1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r}.$$

(Note that $(1-x)(1+x+x^2+\cdots+x^n) = 1-x^{n+1}$ and we divided both sides by $1-x$.) Multiplying it by a, we have the desired. □

Theorem 1.4 (Sum of Infinite Geometric Series)
The sum of an infinite geometric series with first term a and common ratio r **such that** $|r| < 1$ is

$$\frac{a}{1-r}.$$

Proof. Like all previous proofs, lets write out the desired sum. It is

$$a + ar + ar^2 + \cdots .$$

Lets let this value be S. Now, we multiply this by r and subtract it from the original equation. Our equation is now

$$S - S\cdot r = (a + ar + ar^2 + \cdots) - (ar + ar^2 + ar^3 + \cdots).$$

In the right hand side, all the terms cancel out except a. Since we want to isolate S, we factor S from the left hand side and divide by $1-r$. We have

$$S = \boxed{\frac{a}{1-r}}$$

as desired. Note that we have the condition that $|r| < 1$ or else the sum approaches infinity or negative infinity and is divergent (you will learn what that is later). □

§2 Introduction

A summation is denoted the by the sigma symbol, which looks like $\sum$. A summation can be thought of as the sum of a list of numbers. For example, expressing $1+2+3+4+5$ in summation notation would look like

$$\sum_{n=1}^{5} n.$$

In this handout, we use some terms that come up very often and are important to know and understand. These terms can help you with more complicated summations which we will be getting into later.

Definition 2.1. The **dummy variable** in a summation is the variable we are using to evaluate our sequence. In the example shown above, the dummy variable is n.

Definition 2.2. The **index** of a summation is the range of values it evaluates. For example, we evaluated n for terms $1 \to 5$ or $1, 2, 3, 4, 5$. The index maps over $\mathbb{Z}$, or the integers. This means you can't start or end at values like 2.3, π, $\ldots$.

Definition 2.3. The **function** we want to evaluate can be changed. In the example, we summed the values of n from 1 to 5. If we change the summation to

$$\sum_{n=1}^{5} 2n-1,$$

we would now sum the values of $2n-1$ from 1 to 5.

Definition 2.4. A standard summation can be expressed as

$$\sum_{k=n}^{m} f(k)$$

and the value of this can be written as

$$f(n)+f(n+1)+\cdots+f(m-1)+f(m).$$

§3 Evaluating Basic Summations

In this section, we will be learning how to evaluate summations. These are usually just found by adding them manually or using formulas and tricks to evaluate these.

Let's start with a easy example.

Example 3.1

Find $\sum\limits_{n=1}^{7} n$.

Solution. The sum of the first n numbers is $\frac{n(n+1)}{2}$. Plugging in $n=7$, we get $\boxed{28}$. □

Here is a slightly harder one.

Example 3.2

Evaluate $\sum\limits_{n=0}^{\infty} \frac{1}{3^n}$.

Solution. Writing down the first few terms of this infinite summation, we realize that this forms a geometric series. The formula for the sum of an infinite geometric series with first term a and common ratio r is $\frac{a}{1-r}$. We realize this has first term 1 and common ratio $\frac{1}{3}$. Plugging it in, we get $\boxed{\frac{3}{2}}$. □

Example 3.3

Evaluate $\sum\limits_{n=1}^{\infty} 2^n$.

Solution. (Bogus Solution) As this is an infinite geometric series with first term 2 and common ratio 2, the sum is $\frac{2}{1-2}$ or $\boxed{-2}$. □

The Bogus Solutions forgets that the formula for the sum of an infinite geometric series only works if the common ratio, r, has an absolute value less than 1. Since $|2| > 1$, this is not defined. The value of this summation approaches infinity. We have a special name for these summations.

3.1 Classifying Summations

Definition 3.4. A summation that approaches the value infinity or negative infinity is called a **divergent series**.

There is also a name for series that do evaluate to a certain value. These are summations like 2.1 and 2.2.

Definition 3.5. A series or summation that is not divergent is called a **convergent series**.

3.2 Basic Properties

Lets take a look at some properties that could help make summations easier to solve.

Theorem 3.1 (Associative Property of Summations)

$$\sum_{k=n}^{m} f(k) + p(k) = \sum_{k=n}^{m} f(k) + \sum_{k=n}^{m} p(k)$$

Theorem 3.2 (Distributive Property of Summations)

$$\sum_{k=n}^{m} pf(k) = p\sum_{k=n}^{m} f(k)$$

These two properties can help a lot when trying to break apart more complex summations into easier ones that we can find the value of. Lets put these in use.

Example 3.6

Find $\sum_{i=1}^{100} 4 + 3i$.

Solution. Using the two theorems we just learned, we can break apart this summation into two summations. We get

$$\sum_{i=1}^{100} 4 + 3 \cdot \sum_{i=1}^{100} i.$$

The first summation is just adding 4 one hundred times so that value equates to 4×100 or 400. For the second summation, the sum of the first n numbers is $n(n+1)/2$ and when $n = 100$, we get 5050. Our answer is

$$400 + 5050 \times 3 = 400 + 15150 = \boxed{15550}.$$

□

3.3 Exercises

Exercise 3.7. Compute $\sum_{k=7}^{12} 3^k$.

Exercise 3.8. Compute $\sum_{i=1}^{200} (i-3)^2$.

Exercise 3.9. Evaluate $\sum_{i=1}^{9} \binom{10}{i}$.

4 Special Summations

This section is about a list of, how should I say it, *interesting* summations. We won't be proving them here, but you can try them as an extra challenge. Some of these can be pretty useful, while some are just for your own curiosity.

Theorem 4.1 (Sum of First K Squares)

$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}$$

Theorem 4.2 (Sum of First K Cubes)

$$\sum_{n=1}^{k} n^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Theorem 4.3 (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Theorem 4.4 (Nth Row of Pascal's Triangle)

$$\sum_{m=0}^{k} \binom{k}{m} = 2^k$$

Corollary 4.1 (Sum of All Positive Integers)

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}$$

5 Shifting the Index

Shifting the index is a very useful technique while trying to solve summations. This involves changing, or shifting, the index as the name suggests. Lets try a problem using this technique.

Definition 5.1. Shifting the index involves changing the range that the index maps over. This is usually done to make it easier to evaluate. If the problem involves complicated operations on the dummy variable, changing the index can make it easier. You change it such that it ranges over the same amount of numbers but the start point and end point change by the same value. To get a better understanding, lets look at an example.

Example 5.2

Find the value of $\sum_{n=13}^{27} 2^{n-4} \cdot 3$.

Solution. Its a little hard to evaluate this summation especially with $n-4$ in the exponent. This is where shifting the index comes handy. We can change the index that is $13 \to 27$ to $9 \to 23$. In doing this, we can rewrite the summation to

$$\sum_{n=9}^{23} 2^n \cdot 3.$$

IMPORTANT: You need to change **all** the occurrences of n so that it matches the new index.

We can now take out the 3 and use the sum of a finite geometric series formula to get 16776704×3 or $\boxed{50330112}$. $\square$

Example 5.3

A customer wants to buy a pyramid from you. The pyramid has 14 layers, where the first layer is the topmost. For the n^{th} layer, there are n^2 blocks. The customer pays \$11 for any inner block and \$7 for any outer block. Consider the one block on the top to be an outer one. Find the cost of the pyramid.

Solution. Lets try to find a general formula for the number of inner and outer blocks on the n^{th} layer. The number of outer blocks is the perimeter subtracted by 4, as each of the corners are counted twice. The number of inner blocks is a square, with side length $n-2$. For the n^{th} layer, we have $4(n-1)$ outer block and $(n-2)^2$.

We can also ignore the 1st layer, as $n-2$ will go below 0. We write this as a summation

$$\sum_{n=2}^{14} 11 \cdot (n-2)^2 + 7 \cdot 4(n-1).$$

We use the two properties we just saw in the last section to break this up into

$$11 \sum_{n=2}^{14} (n-2)^2 + 28 \sum_{n=2}^{14} n-1.$$

This is where shifting the index comes in handy. We change the index of the first summation to $0 \to 12$, so the value we want to calculate becomes n^2. For the second summation, we change the index to $1 \to 13$, so the value we want to calculate is n. Rewritten, we have

$$11 \sum_{n=0}^{12} n^2 + 28 \sum_{n=1}^{13} n.$$

The sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$, and with $n = 12$, we get 650. The sum of the first n integers is $\frac{n(n+1)}{2}$ and with $n = 13$, we get 91. Our answer is

$$650 \cdot 11 + 28 \cdot 91 + 7(\text{The topmost block}) = \boxed{\$9705}.$$

□

Especially in more complicated sums, we need to combine this tool with another one, which we call telescoping.

§6 Telescoping

Telescoping summations are usually summations that look like they diverge, meaning they never reach a finite value, but actually converge. They usually have a bunch of terms that cancel out leaving only a few terms which are easy to calculate. Lets take a look at one of these telescoping sums.

Example 6.1

Evaluate $\sum\limits_{i=1}^{99} \frac{1}{i} - \frac{1}{i+1}$.

Solution. We can write out the first few terms of this sequence to try to find terms that cancel. We find

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{98}-\frac{1}{99}\right)+\left(\frac{1}{99}-\frac{1}{100}\right).$$

All the negative terms cancel out the positive ones, leaving only the first and last term of the sequence. This is $\frac{1}{1}-\frac{1}{100}$ or $\boxed{\dfrac{99}{100}}$. □

Lets look at another way to break down these summations.

6.1 Partial Fraction Decomposition

Definition 6.2. We call the **partial fraction decomposition** of a fraction as breaking down the fraction into smaller ones which are easier to calculate and cancel out terms.

How do we do these and what are they? Lets take a look.

Example 6.3

Find values A and B such that $\frac{2k+3}{k(k+3)}=\frac{A}{k}+\frac{B}{k+3}$.

Solution. At first sight, the fractions look quite ugly. We can get rid of them by multiplying both sides by $k(k+3)$. We know have $2k+3=A(k+3)+Bk$. This seems hard to solve, but notice we can get rid of A or B by plugging in $k=-3$ and $k=0$ respectively. Using $k=-3$, you get $-3=-3B$ which means $B=1$. Plugging in $k=0$, we have $3=3A$ which gives us $A=1$. We are done. □

How do we use these in summations? The trick is to break apart these denominators into two easier ones. Once you do that and write out the first few terms in the series, many terms will cancel with one another. Lets take a look at a familiar example.

Example 6.4

Evaluate $\sum\limits_{n=1}^{99}\frac{1}{n(n+1)}$.

Solution. We look at the denominator and see the terms n and $n+1$. For this fraction's partial fraction decomposition, we set it to equal

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1},$$

where A and B are constants. To find them, we need to manipulate this equation. First things first, we get rid of the fractions by multiplying both sides by $n(n+1)$. We now have

$$1=A(n+1)+Bn.$$

Notice that we can find the two values by plugging in values of n which make parts of the equation 0. To make $A(n+1)=0$, we can let $n=-1$. Plugging this in, we get that $1=-B$ meaning that $B=-1$. To make $Bn=0$, we let $n=0$. Repeating this process, we get that $A = 1$. We know have our partial fraction decomposition. Rewriting our summation, we get

$$\sum_{n=1}^{99} \frac{1}{n} - \frac{1}{n+1}.$$

Notice this is exactly the same as 4.1! Using the same technique as 4.1, we get $\boxed{\frac{99}{100}}$. □

Some summations involve taking a look at fractions with infinitely large denominators. Lets take a look at one of these.

Example 6.5

Find the value of $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+2)}$.

Solution. We try to break this up into its partial fraction decomposition. We set it to equal

$$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2}.$$

Cross multiply to get $1 = A(k+2) + Bk$. We use the values $k = 0$ and $k = -2$. With $k = 0$, we get $2A = 1$ or $A = \frac{1}{2}$. With $k = -2$, we get $-2B = 1$ or $B = -\frac{1}{2}$. We can remove the $\frac{1}{2}$ to make the summation easier to evaluate. We now have

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+2}.$$

We write out a few terms

$$\frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\cdots\right]$$

and find they all cancel except for the first 2 and last 2. The last two have denominators infinitely large so their value approaches 0. The first two terms add up to $\frac{3}{2}$, when multiplied by $\frac{1}{2}$ give us $\boxed{\frac{3}{4}}$. □

Sometimes, partial fraction decomposition gets a little more complicated. This is when there are double roots in the denominator. Lets look at an example.

Example 6.6

Find $A+B+C+D$ if A, B, C, and D are the coefficients of the partial fractions expansion of

$$12 \cdot \frac{x^3+4}{(x^2-1)(x^2+3x+2)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

Solution. We first factor the denominator of the left hand side. Difference of squares gives us $x^2-1 = (x+1)(x-1)$ and factoring x^2+3x+2 gives us $(x+2)(x+1)$. Rewriting the equation, we have

$$12 \cdot \frac{x^3+4}{(x+1)^2(x+2)(x-1)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

Getting rid of the denominators by multiplying both sides by $(x+1)^2(x+2)(x-1)$, we get

$$12(x^3+4) = A(x+2)(x+1)^2 + B(x-1)(x+1)^2 + C(x-1)(x+2)(x+1) + D(x-1)(x+2).$$

We start by using some values of x to cancel out the others.

- Letting $x = 1$, we get that $12 \cdot 5 = A \cdot 3 \cdot 2^2 \to A = 5$.
- Letting $x = -2$, we have $12 \cdot -4 = B \cdot -3 \cdot (-1)^2 \to B = 16$.
- Letting $x = -1$, we get $12 \cdot 3 = D \cdot -2 \cdot 1 \to D = -18$.

We need C. Any value of x that cancels out C will cancel out the rest. This is why we can use the rest of the variables' value to calculate it. Observing the terms with degree 3, we get

$$12x^3 = Ax^3 + Bx^3 + Cx^3.$$

Solving for C, we have $12 = 5 + 16 + C \to C = -9$. The question asks for the sum so we put $5 + 16 - 9 - 18$ or $\boxed{-6}$.

□

Example 6.7 (Alcumus)

Compute

$$\sum_{n=1}^{\infty} \frac{F_{n+1}}{F_n F_{n+2}},$$

where F_n denotes the nth Fibonacci number, so $F_0 = 0$ and $F_1 = 1$.

Solution. [Credits to Alcumus] Since $F_{n+1} = F_{n+2} - F_n$,

$$\frac{F_{n+1}}{F_n F_{n+2}} = \frac{F_{n+2} - F_n}{F_n F_{n+2}} = \frac{1}{F_n} - \frac{1}{F_{n+2}}.$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} &= \left(\frac{1}{F_1} - \frac{1}{F_3}\right) + \left(\frac{1}{F_2} - \frac{1}{F_4}\right) + \left(\frac{1}{F_3} - \frac{1}{F_5}\right) + \cdots \\ &= \frac{1}{F_1} + \frac{1}{F_2} \\ &= \boxed{2}. \end{aligned}$$

□

Example 6.8 (Alcumus)

Compute

$$\frac{5}{3^2 \cdot 7^2} + \frac{9}{7^2 \cdot 11^2} + \frac{13}{11^2 \cdot 15^2} + \cdots.$$

Solution. Using the pattern given, we can write the n^{th} term as

$$\frac{4n+1}{(4n-1)^2(4n+3)^2}.$$

We notice that the value $(4n+3)^2-(4n-1)^2$, by difference of squares, can be written as $8(4n+1)$. So we can rewrite the numerator of the n^{th} term using this. We now have

$$\frac{1}{8}\left[\frac{(4n+3)^2-(4n-1)^2}{(4n-1)^2(4n+3)^2}\right].$$

Simplifying this even further, we now have

$$\frac{1}{8}\left(\frac{1}{(4n-1)^2}-\frac{1}{(4n+3)^2}\right).$$

This seems like it will telescope, and when we actually write out the first few terms, our guess is correct. It starts off as

$$\frac{1}{8}\left(\frac{1}{3^2}-\frac{1}{7^2}\right)+\frac{1}{8}\left(\frac{1}{7^2}-\frac{1}{11^2}\right)+\frac{1}{8}\left(\frac{1}{11^2}-\frac{1}{15^2}\right)+\cdots.$$

Notice that all the terms cancel out except the first and last one, and since the last one is equivalent to 0, our answer is $\frac{1}{8}\cdot\frac{1}{3^2}$ or $\boxed{\frac{1}{72}}$. □

Here's a significantly hard problem, where you need to add intuition into your telescoping.

Example 6.9 (1995 AIME)

Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Find $\sum_{k=1}^{1995}\frac{1}{f(k)}$.

Solution. Let's find how many times a value n appears. We need it to be in between $\left(n-\frac{1}{2}\right)^4$ and $\left(n+\frac{1}{2}\right)^4$. Well, the number of values in this range is just

$$\left\lfloor\left(n+\frac{1}{2}\right)^4-\left(n-\frac{1}{2}\right)^4\right\rfloor.$$

Well, we can expand this using the binomial theorem. Let's just ignore the floors for right now (and we'll see we didn't even need them!). We get

$$\left(n^4+2n^3+\frac{3}{2}n^2+\frac{1}{2}n+\frac{1}{16}\right)-\left(n^4-2n^3+\frac{3}{2}n^2-\frac{1}{2}n+\frac{1}{16}\right)=4n^3+n.$$

So we know that the value $\frac{1}{n}$ appears $4n^3+n$ times. The total value for these occurrences is $4n^2+1$. Notice that the highest value of n is 7, but some values m that $f(m)=7$ are not in the range given. We can just compute $4n^2+1$ for $1\le n\le 6$, and then add the values of 7 at the end. We know that

$$\sum_{n=1}^{6}4n^2+1=370,$$

by the sum of squares formula. What is the highest value m such that $f(m)=6$? Well, if we expand $\left(6+\frac{1}{2}\right)^4$ by the binomial theorem we get it to be 1785.0625 meaning $m=1785$. There are $1995-1785=210$ values of m such that $f(m)=7$ meaning $\frac{1}{7}\cdot 210=30$ is added to 370, giving us $\boxed{400}$. □

6.2 Exercises

Exercise 6.10 (Unknown). Evaluate the following sum $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\dots+\sqrt{1+\frac{1}{2011^2}+\frac{1}{2012^2}}$.

Exercise 6.11 (Brilliant). Evaluate

$$\frac{1}{\sqrt{4}+\sqrt{7}}+\frac{1}{\sqrt{7}+\sqrt{10}}+\dots+\frac{1}{\sqrt{397}+\sqrt{400}}$$

Exercise 6.12 (2002 AIME). Find the integer that is closest to $1000\sum\limits_{n=3}^{10000}\frac{1}{n^2-4}$.

Exercise 6.13 (2019 Duke Math Meet). Let $A=\frac{4}{1\cdot2\cdot3}+\frac{5}{2\cdot3\cdot4}+\frac{6}{3\cdot4\cdot5}+\dots+\frac{101}{98\cdot99\cdot100}$. If $B=\frac{5}{4}-A$, then when written in lowest terms, $B=\frac{p}{q}$. Find $p+q$

Exercise 6.14 (2017 CMC). For every positive integer n, we define the operation $n^\circ=n\cdot n!$. What is the remainder when $1^\circ+2^\circ+\dots+17^\circ$ is divided by 23?

7 Nested Summations

In many problems, we have nested summations. These are summations that are put inside of one another. For nested sums, try to solve the inner one first and then move on to the outer one. Lets look at an easy one.

Example 7.1

Find $\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{n} jk$ in terms of n.

Solution. Notice that we can distribute out a j from the inner sum. We now get $\sum\limits_{j=1}^{n} j\sum\limits_{k=1}^{n} k$. The inner sum is just $\frac{n(n+1)}{2}$. We can factor this out of the outer sum. We have

$$\frac{n(n+1)}{2}\sum_{j=1}^{n} j=\boxed{\left[\frac{n(n+1)}{2}\right]^2}$$

□

Example 7.2

In terms of n, find $\sum\limits_{i=0}^{n}\sum\limits_{k=i+1}^{n} 4$.

Solution. Lets solve the inner summation first. Using Theorem 3.2, we can rewrite it as $4\sum\limits_{k=i+1}^{n} 1$. How do we do this? Essentially, the summation is asking how many terms we evaluate as we just add 1 every time. This is basically number of integers between $i+1$ and n inclusive, and the value of that is $n-(i+1)+1$ or $n-i$. Our simplified sum is

$$\sum_{i=0}^{n} 4(n-i).$$

Repeating the Distributive property, we have $4\sum_{i=0}^{n} n-i$. Since n is a fixed value, lets just test a few values to try to find a pattern. If $n=1$, the summation is $4(1+0)$. If $n=2$, the summation is $4(2+1+0)$. If $n=3$, the summation is $4(3+2+1+0)$. We start to see a pattern here. It is just the sum of the first n natural numbers multiplied by 4. The formula for that is $\frac{n(n+1)}{2}$ and multiplying it by 4, we get

$$\boxed{2n(n+1)}.$$

□

Theorem 7.1 (Swapping the Order of Summations)

Define function $p(a,b)$. We always have

$$\sum_{a\in A}\sum_{b\in B} p = \sum_{b\in B}\sum_{a\in A} p.$$

Note: The notation just is an easier way to express a standard summation.

Lets look at a problem where this comes in handy.

Example 7.3 (Classic)

Evaluate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{1}{2^{m+n}}.$$

Solution. If we try to evaluate the summation as it is currently, we have to use a finite geometric series formula which eventually gets a bit too messy. This is where we switch the order. What values of n work? Well, n has to be greater than or equal than 1 but less than m. m can be anything. We can bound these values by this inequality:

$$1 \le n \le m \le \infty.$$

To switch the order, we want the same inequality to be satisfied. So if we let the first summation be $\sum\limits_{n=1}^{\infty}$, what should we make the second sum? Since m has to be greater than or equal to n, we can let the second sum be $\sum\limits_{m=n}^{\infty}$. Our new summation is

$$\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{1}{2^{m+n}}.$$

We can expand $\frac{1}{2^{m+n}}$ to be $\frac{1}{2^m}\cdot\frac{1}{2^n}$. We can then factor out the $\frac{1}{2^n}$. Our new summation is

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\sum_{m=n}^{\infty}\frac{1}{2^m}.$$

Notice the inner sum is just an infinite geometric series with first term $\frac{1}{2^n}$ and common ratio $\frac{1}{2}$. The sum of this is $\frac{\frac{1}{2^n}}{1-\frac{1}{2}}$. Plugging this in and simplifying, we now only have one sum to evaluate, which is

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{2}{2^n} \to \sum_{n=1}^{\infty}\frac{2}{4^n} \to 2\sum_{n=1}^{\infty}\frac{1}{4^n}.$$

The sum is just an infinite geometric series with first term $\frac{1}{4}$ and common ratio $\frac{1}{4}$. The sum of this is just $\frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$. Multiplying by 2, we get our answer, $\boxed{\frac{2}{3}}$. □

Lets look at one last problem.

Example 7.4 (HMMT 2008)
Compute

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{k}{2^{n+k}}.$$

Solution. It's hard to evaluate the current summation, so let's try switching the order. Again, lets try to find the range of values n and k that work. We get the inequality $1 \le k \le n-1 \le \infty$. Notice that we can rewrite this as $1 \le k+1 \le n \le \infty$. This means the summation can be written as

$$\sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty}\frac{k}{2^{n+k}}.$$

We can factor out the $\frac{k}{2^k}$ since k is a constant value. The inner sum is $\sum\limits_{n=k+1}^{\infty}\frac{1}{2^n}$ which, by the infinite geometric series formula, is $\frac{1}{2^k}$. We now have

$$\sum_{k=1}^{\infty}\frac{k}{4^k}.$$

Let S be the desired sum, such that

$$S = \frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \dots.$$

Now, if we divide this by 4 and subtract it from S we get

$$S - \frac{S}{4} = \frac{1}{4} + \left(\frac{2}{4^2} - \frac{1}{4^2}\right) + \left(\frac{3}{4^3} - \frac{2}{4^3}\right) + \dots = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3} \implies S = \boxed{\frac{4}{9}}.$$

□

§8 Problems

Exercise 8.1 (1997 AIME)**.** The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number s/r can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

Exercise 8.2 (HEOMT Individual)**.** Evaluate the following sum.

$$\sum_{n=1}^{30}\frac{62-2n}{n(n+1)} + \frac{2}{32-n}$$

Exercise 8.3 (2002 AIME)**.** Consider the sequence defined by $a_k = \dfrac{1}{k^2+k}$ for $k \ge 1$. Given that $a_m + a_{m+1} + \cdots + a_{n-1} = \dfrac{1}{29}$, for positive integers m and n with $m < n$, find $m+n$.

Exercise 8.4 (2013 HMMT)**.** Compute the prime factorization of 1007021035035021007001.

Exercise 8.5 (Stanford 2011)**.** Find the value of the sum

$$\sum_{n=1}^{\infty} \frac{(7n+32)\cdot 3^n}{n(n+2)\cdot 4^n}.$$

Exercise 8.6 (BMT Spring 2019)**.** Let (k_i) be a sequence of unique nonzero integers such that $x^2 - 5x + k_i$ has rational solutions. Find the minimum possible value of

$$\frac{1}{5}\sum_{i=1}^{\infty} \frac{1}{k_i}.$$

Exercise 8.7 (Intermediate Algebra)**.** Evaluate

$$\sum_{k=1}^{7} \frac{1}{\sqrt[3]{k^2} + \sqrt[3]{k(k+1)} + \sqrt[3]{(k+1)^2}}.$$

Exercise 8.8. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all integers he wrote down.
Hint: Find the amount of times he writes a certain number.