Induction

Rohan Garg

May 2021

Contents

1 Introduction

What is induction? Induction is a type of proof used in many problems. The idea behind the proof is like a set of dominoes.

You first prove that it works for what we call the *base case*, which is the first, or sometimes smallest, value that the problem asks for. Next, you perform the *inductive step*. You assume that it works for n, and prove that it works for $n+1$. Since you proved the base case, you proved $n+1$, and by proving $n+1$, you proved $n+2$, and this continues forever.

Definition 1.1. Finding and showing that the first value possible works is called the *base case*.

Definition 1.2. Assuming n is true, showing that $n+1$ is also true is called the *inductive step*.

1.1 Proof Examples

Here is how a basic proof would look like.

> **Example 1.3**
> The base case is *base case here*. It works because *show the base case works here*. Now we must show the inductive step. We know that it works for *the original formula*. We show that it works for $n+1$. *Prove the inductive step here.* We are done.

2 Engineer's Induction

Engineer's Induction is not a valid proof. Why are we learning about this though? It is actually very helpful in short answer contests. The idea behind it is very similar to induction, except you remove all the "proof" part of it. You have to test out small values and find a pattern, assume it's true, find the answer to the problem using that pattern.

Example 2.1 (2021 AMC 10)

Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

Solution. Lets try to compute the first few values multiplied. If we just look at the first term, its $2+3=5$. If we look at the first and second, we get $5\cdot 13=65$. Examining the answer choices, we notice that these form a pattern, that $3^2-2^2=5$ and $3^4-2^4=65$. By Engineer's Induction, our answer for the problem should be $3^{128}-2^{128}$ or $\boxed{\text{C}}$. □

Let's look at one more problem, which is significantly hard without this.

Example 2.2 (2021 ARML Local)

Define a sequence as $a_1=x$ for some real number x and

$$a_n = na_{n-1}+(n-1)(n!(n-1)!-1)$$

for integers $n\geq 2$. Given that $a_{2021}=(2021!+1)^2+2020!$, and given that $x=\frac{p}{q}$, where p and q are positive integers whose greatest common divisor is 1, compute $p+q$.

Solution. Let's try to find a_2, a_3, a_4 in terms of a_1. a_2 will be $2x+(2-1)(2!\cdot 1!-1)=2x+1$, a_3 will be $3(2x+1)+(3-1)(3!\cdot 2!-1)=6x+25$, and a_4 will be $4(6x+25)+(4-1)(4!\cdot 3!-1)=24x+529$.

Notice that the coefficient of x in the term a_n is $n!$. The constant term is a perfect square, and we notice it is 0^2 for a_1, 1^2 for a_2, 5^2 for a_3, and 23^2 for a_4. These are all 1 less than $n!$. So we claim that $a_n=n!\cdot x+(n!-1)^2$. Now we just need to solve for x, because we know $a_{2021}=2021!x+(2021!-1)^2$. We get

$$2021!x+(2021!-1)^2=(2021!+1)^2+2020! \implies (2021!+1)^2-(2021!-1)^2=2021!x-2020!.$$

Notice the left hand side is just $4\times 2021!$, which you get when you expand. We can also factor out a 2020! from the right hand side, to get

$$4\times 2021! = 2020!(2021x-1).$$

We can divide both sides by 2020! to get

$$4\cdot 2021 = 2021x-1 \implies 2021x=8085 \implies x=\frac{8085}{2021} \implies 8085+2021 \implies \boxed{10106}.$$

□

2.1 Exercises

Exercise 2.3 (NICE Spring 2021)**.** Fifty rooms of a castle are lined in a row. The first room contains 100 knights, while the remaining 49 rooms contain one knight each. These knights wish to escape the castle by breaking the barriers between consecutive rooms, ending with the barrier from room 50 to the outside.

At the stroke of midnight, each knight in the i^{th} room begins breaking the barrier between the i^{th} and $(i+1)^{\text{st}}$ rooms, where we count the 51^{st} room as the exterior. Each person works at a constant rate and is able to break down a barrier in 1 hour, and once a group of knights breaks down the i^{th} barrier, they immediately join the knight breaking down the $(i+1)^{\text{st}}$ barrier.

The number of hours it takes for the knights to escape the castle is $\frac{m}{n}$, where m and n are positive relatively prime integers. Compute the product mn.

Exercise 2.4 (PUMaC 2018)**.** If $a_1, a_2, \dots$ is a sequence of real numbers such that for all n, $\sum_{k=1}^{n} a_k \left(\frac{k}{n}\right)^2 = 1$, find the smallest n such that $a_n < \frac{1}{2018}$.

Exercise 2.5 (OMO 2018)**.** A mouse has a wheel of cheese which is cut into 2018 slices. The mouse also has a 2019-sided die, with faces labeled $0, 1, 2, \dots, 2018$, and with each face equally likely to come up. Every second, the mouse rolls the dice. If the dice lands on k, and the mouse has at least k slices of cheese remaining, then the mouse eats k slices of cheese; otherwise, the mouse does nothing. What is the expected number of seconds until all the cheese is gone?

§3 Basic Examples of Induction

In this section, we will go over basic examples of induction which require just a simple understanding of the definition. Some of these identities might be familiar from our summations handout.

Example 3.1 (Sum of First N Positive Integers)

Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. The base case is $n = 1$. We know that $\frac{1(1+1)}{2} = 1$, which means the base case is true. Now we must perform the inductive step. We assume it's true for n, and prove that it works for $n + 1$. We are provided

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Adding $n + 1$ to both sides, we get

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + n + 1 \implies \frac{n^2 + n + 2n + 2}{2} \implies \frac{(n+1)(n+2)}{2},$$

which completes our inductive step since by the formula, it should be $\frac{(n+1)(n+1+1)}{2}$, which is what we got. □

Let's look at one more example before we get into more complex induction.

Example 3.2

Prove that for any positive integer number n, $n^3 + 2n$ is divisible by 3.

Proof. The base case is $n = 1$. Since $1^3 + 2 \cdot 1 = 3$, the base case is true. Now we prove the inductive step. We know that for $n + 1$ the expression will be $(n + 1)^3 + 2(n + 1) = n^3 + 3n^2 + 5n + 3$. Notice that we assumed $n^3 + 2n$ is divisible by 3, so we can write the expression as

$$(n^3 + 2n) + (3n^2 + 3n + 3) \implies (n^3 + 2n) + 3(n^2 + n + 1).$$

We are done, as the expression $3(n^2 + n + 1)$ is a multiple of 3 and we already know $n^3 + 2n$ is divisible by 3. □

4 More Induction

In this section, we will examine different types of induction that are used in different problems.

4.1 Base Case Is Not 1

Many induction problems have cases where the base case is not 1. In these, we perform the same steps except the base case is different. Let's look at one example.

Example 4.1

Prove that $n^2 < 2^n$ for sufficiently large values n.

Proof. What is a "sufficiently large" value of n? It means that after we reach the least value of n that works, every value after that will work as well. If we just try values, we get the least value of n to be 5. That is our base case. The base case is true, since $32 > 25$. Now we perform the inductive step. We need to show

$$(n + 1)^2 < 2^{n+1}.$$

Expanding $(n + 1)^2$, we get $n^2 + 2n + 1$. Then, we can write the inequality

$$n^2+2n+1 < n^2+2n+n \implies n^2+2n+1 < n^2+3n \implies n^2+2n+1 < n^2+n^2 \implies n^2+2n+1 < 2n^2.$$

If you don't understand how we did that, notice that $n \geq 5$ so $n > 1$ and $n^2 > 3n$. Now, we can use the inductive step, which says $n^2 < 2^n$ and substitute this into our inequality to get

$$n^2 + 2n + 1 < 2(2^n) \implies n^2 + 2n + 1 < 2^{n+1} \implies (n + 1)^2 < 2^{n+1},$$

as desired. □

4.2 Strong Induction

Strong induction is a very useful type of induction. What is it? Strong induction is similar to normal induction. Here is the process.

1. Show the base case is true.
2. This is where strong induction differs from normal induction. We assume its true for ALL values from 1 to k to prove it is true for $k + 1$.

Let's look at an example.

Example 4.2 (Fundamental Theorem of Arithmetic)

Every integer $n \geq 2$ can be written uniquely as the product of prime numbers.

Proof. The base case is $n = 2$. This is prime, so it is true. We assume it is true for $2, 3, 4, \ldots, n$. If $n + 1$ is prime, we are already done. Else, we can express it as pq, where $1 < p, q < n$. We assumed every value between 2 and n works, so p and q can be expressed as a product of prime numbers. Thus, pq, or n, can be expressed uniquely as the product of prime numbers. □

This was a quite simple example with strong induction, but many harder problems can be solved using this. Let's try one, using another type of induction.

4.2.1 More than one base case (Strong Induction)

The following problem will require you to prove multiple base cases. This is a useful technique.

Example 4.3 (USAJMO 2011)

A word is defined as any finite string of letters. A word is a palindrome if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, ...$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \geq 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \geq 1$, the word formed by writing $W_1, W_2, W_3, ..., W_n$ in succession is a palindrome.

Proof. The base case for this proof is W_1 and W_1W_2. These words are b and bab respectively, which are both palindromes. We proceed with the inductive step. Notice that $W_n = W_{n-2}W_{n-1}$, so we need to show

$$W_1W_2 \ldots W_{n-1}W_{n-2}W_{n-1}$$

is a palindrome. If this is a palindrome, we can reverse it and show that the reversed word is the palindrome. The reversed word is $W_{n-1}W_{n-2}W_{n-1} \ldots W_2W_1$. We also know by the inductive step that $W_1W_2 \ldots W_{n-1}$ is a palindrome so we can reverse it in our word to get

$$W_{n-1}W_{n-2}W_1W_2 \ldots W_{n-1}.$$

The inductive step also gives us $W_1W_2 \ldots W_{n-3}$ is a palindrome so we can reverse that to get

$$W_{n-1}W_{n-2}W_{n-3}W_{n-4} \ldots W_2W_1W_{n-2}W_{n-1}.$$

We can reverse $W_{n-1}W_{n-2} \ldots W_2W_1$ to get

$$W_1W_2 \ldots W_{n-1}W_{n-2}W_{n-1}.$$

Notice that this is the same expression we started with, since $W_{n-2}W_{n-1}$ is W_n. From the inductive step, we showed that $W_nW_{n-1} \ldots W_1 = W_1W_2 \ldots W_n$. We are done, since we have completed the inductive step. □

Remark 4.4. The reason we used strong induction here is so we can reverse expressions such as $W_1W_2 \ldots W_{n-3}$ to try to get the original expression, completing the inductive step.

5 Problems

Exercise 5.1 (Sum of first N squares)**.** Prove that $1 + 2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 5.2 (Sum for first N cubes)**.** Prove that $1 + 2 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Exercise 5.3 (Subsets of a Set)**.** Show that a set of n elements has 2^n subsets.

Exercise 5.4. Prove that $n! > 2^n$ for sufficiently large values of n.

Exercise 5.5 (Diagonals of a Polygon)**.** Show that an $n-$sided polygon has $\frac{n(n-3)}{2}$ diagonals.

Exercise 5.6. Consider the Fibonacci sequence where $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for all positive integers n. Prove that

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Exercise 5.7 (Explicit Form for Fibonacci Sequence)**.** The Fibonacci sequence is defined as $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0, F_1 = 1$. Show that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Exercise 5.8. Prove that for any natural number $n \geq 6$, then

$$(n+3)^3 \leq 3^n.$$

Exercise 5.9 (2009 AIME)**.** Let m be the number of solutions in positive integers to the equation $4x + 3y + 2z = 2009$, and let n be the number of solutions in positive integers to the equation $4x + 3y + 2z = 2000$. Find the remainder when $m - n$ is divided by 1000.

Exercise 5.10 (CALT April Fools Contest)**.** Prove that the summation

$$\sum_{a_n=1}^{k} \sum_{a_{n-1}=1}^{a_n} \cdots \sum_{a_2=1}^{a_3} \sum_{a_1=1}^{a_2} a_1$$

is equivalent to

$$\frac{k(k+1)(k+2)\ldots(k+n)}{(n+1)!}.$$

Exercise 5.11 (PiE)**.** Prove that if $(A_i)_{1 \leq i \leq n}$ are finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|.$$

Exercise 5.12 (1981 IMO)**.** Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that
$F(n, r) = \frac{n+1}{r+1}$.