# Counting and Probability Part 1

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## 1 Introduction

**Definition 1.1.** The factorial of a positive integer is the product of all the positive integers less than or equal to it. It is normally denoted as $n!$. For example, $4! = 4 \times 3 \times 2 \times 1 = 24$.

## 2 Distinct vs. Indistinct

In these questions, it's important to understand what distinct and indistinct objects mean in combo problems.

- If two distinct objects switch places, the result would count as a different arrangement.
  - ∗ People with different names are almost always assumed as indistinct.
- If two indistinct objects switch places, the result would count as the same arrangement.
  - ∗ Common nouns such as items, fruits, etc are almost always considered indistinct.
- "Distinct" objects are sometimes called "distinguishable" and "indistinct" objects are sometimes called "indistinguishable".

Suppose you are solving a counting problem involving a shape, and you see the phrase "rotations are considered distinct/indistinct" or "reflections are considered distinct/indistinct" at the end of a problem. Here's what that means.

- "Rotations are indistinct" means if you can rotate one arrangement to look like another, then those arrangements count as the same thing. If rotations are distinct, then they count as different.
- "Reflections are distinct/indistinct" is defined similarly.

## 3 Combinations

**Example 3.1**

I have 10 puppies. I want to take 3 of them on a walk. How many ways can I do that?

*Solution.* We can pick the first puppy in 10 ways, the second in 9 ways, and the third in 8 ways. This gives us $10 \times 9 \times 8 = 720$ ways. Is this our final answer? No, because I do not care about the order of the puppies. Any set of 3 puppies can be arranged in 3! or 6 ways, as the first puppy can be in 3 spots, the second can be in 2 spots, and the last has only 1 spot. This means our final answer is $\frac{720}{6} = 120$. □

One of they key takeaways is that order does not matter in combinations. That is the reason we had to divide by 6 at the end. Is there a general formula?

**Example 3.2**

How many ways can I choose $k$ objects out of a group of $n$ objects?

*Solution.* We have to repeat the same process we did above. For the first object, there are $n$ ways, for the second object there are $n-1$ ways, all the way to $n-k+1$ for the $k$th object (note that it's $n-k+1$, not $n-k$). This value can be written as $\frac{n!}{(n-k)!}$. Notice that when we pick out the $k$ objects, the set of the $k$ objects doesn't matter, so we have to divide by $k!$ because we are over counting for each way that we can arrange the items in the set we chose. So the formula for Combinations is

$$\frac{n!}{k!(n-k)!}$$

□

**Theorem 3.1** (Combinations)

The number of ways we can choose $k$ objects out of a group of $n$ objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Its important to understand why this formula works, not necessarily to memorize it.

**Example 3.3**

There are 9 distinct chairs. How many ways are there to group these chairs into 3 groups of 3?

*Solution.* For the sake of simplicity, lets assume the order of the groups matter. The first group can be chosen in $\binom{9}{3} = 84$ ways. The second group can be chosen in $\binom{9-3}{3} = \binom{6}{3} = 20$ ways. The last group is just the remaining 3 people, so there is only 1 way to form that group. Our answer should be $84 \times 20$, but we forgot that in this answer, we assumed the order of the groups matter. To get rid of this, we divide by 3! which is the number of ways to order the groups. Our final answer is

$$\frac{84 \times 20}{3!} = \boxed{280}.$$

□

Lets look at a harder problem.

**Example 3.4**

Find the number of rectangles in a $p \times q$ grid.

*Solution.* We don't know where to start, so we try to see how many ways we can choose the two horizontal and vertical lines for the rectangle. Let's start from a smaller example. If we have a $2 \times 2$ grid, we have a total of 3 vertical and 3 horizontal lines. And what we want to do, is pick 2 of each type of line. So we are essentially doing $\binom{3}{2} \times \binom{3}{2}$. Now if we want to generalize this for $p \times q$ grid, without loss of generality, we know that there is going to be $p+1$ vertical lines, and $q+1$ horizontal lines. Now all we want to do is choose 2 of each type of line. So our final answer is going to be $\boxed{\binom{p+1}{2} \times \binom{q+1}{2}}$. □

## §4 Permutations

**Example 4.1**
How many ways can I give 8 people a gold, a silver, and a bronze medal?

*Solution.* What we learned from the previous section can be applied here. We have to choose 3 people from the 8 people to get the medals. So we can do that and we get $\binom{8}{3} = 56$. From here, we notice that we don't want to chose a group of people to get this, but individuals who will receive the different awards. So in that group we have to multiply by 3! so we assign each individual in there the award they got. Wait but didn't we divide 3! in the combinations? So is there a formula for this? But let's try and get it ourselves. We notice that $\binom{8}{3} = \frac{8!}{3!\times 5!}$ and multiplying 3! gives us $\frac{8!}{5!} = \boxed{336}$ □

**Example 4.2**
How many ways can I choose $k$ people from a group of $n$ people such that the ordering of the people matter?

*Solution.* Notice that the problem is very similar to Example 3.2 except that the ordering of the people matter. So let's start of with a base of $\binom{n}{k}$. Now from here we see that the order of the people matter. So in that group, there is $k$ ways to choose the first person, $k-1$ ways to choose the second person, and so on until we have 1 way to choose the $k$th person. So in total we have $k!$ ways to choose the people. Multiplying this by $\binom{n}{k}$ we get $\frac{n!}{(n-k)!}$ ways in total to achieve this. □

**Theorem 4.1** (Permutations)
The number of ways we can choose $k$ objects out of a group of $n$ objects such that the $k$ objects need to be ordered is

$$P(n,k) = \frac{n!}{(n-k)!}$$

Like we see this, the permutation is written as $P(n,k)$.

Combinations are in turn similar to permutations. $\binom{n}{k}$ is $P(n,k)$ divided by $k!$, which is to compensate for the fact that combinations do not care about the order objects are chosen.

**Example 4.3**
What is $P(7,3)$?

*Solution.* We can plug this into the formula with $n = 7$ and $k = 3$. Plugging in we get

$$\frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = \boxed{210}$$

□

**Example 4.4**

Looking at the same example from before, Jess has 3 medals now and 12 people. She wants to give a gold, a silver, and a bronze medal. How many ways are there for this to happen?

*Solution.* We know that she has to give 3 medals to 12 people. This is a perfect problem to use combinations. We care about the order, so we can let $n = 12$ and $k = 3$. Now using the formula we get:

$$\frac{12!}{9!} \implies 1320$$

□

**Example 4.5**

Suppose Alice has 5 red balls and 4 green balls. She wants to arrange all the balls in a row in such a way that the red balls occupy the odd positions. How many arrangements are possible?

*Solution.* So we have 5 positions for the red balls, and 4 positions for the green balls, so we have $5! \times 4! = \boxed{2880}$ □

**Example 4.6** (Quora)

In how many ways you can arrange 5 rings in your right hand fingers (Not counting your thumb)?

*Solution.* Notice that the first ring can be put on 4 different fingers. The second ring has an option of 5 ways, as we have 4 different fingers, and it can go below or above the first ring. The third ring will have the same options except we have to add 1 more so 6 options. This will go onto the last ring which will have a total of 8 options. Multiplying them all together, we get $4 \times 5 \times 6 \times 7 \times 8 = \boxed{6720}$ □

## 4.1 Arrangment of Groups Between Indistinct Groups

**Example 4.7**

How many ways can we order INDIANA if all letters are indistinguishable?

*Solution.* Notice that in the word INDIANA there are:

- $2 \times A$
- $2 \times I$

- $2 \times N$
- $1 \times D$

Notice that we have 7 letters in total and we want to put 2 $A$s, 2 $I$s, 2 $N$s, and 1 $D$. Out of the 7 options, we need to choose 2 for the A, 2 $I$s of the remaining 5 spots, 2 $N$s of the remaining 3 spots, and we have the last spot for $D$. Writing this as a combination we see this:

$$\binom{7}{2}\binom{5}{2}\binom{3}{2}\binom{1}{1} \implies$$

□

From this we can make a general idea. If we have $a_1$ indistinct terms in group 1, $a_2$ indistinct terms in group 2, ..., $a_n$ indistinct terms in group $n$, the total number of ways to order it is:

$$\frac{(a_1 + a_2 + a_3 \dots a_{n-1} + a_n)!}{a_1! \times a_2! \times a_3! \dots a_{n-1}! \times a_n!}$$

## 5 Combinations vs Permutations

As you can see, permutations and combinations have very subtle difference. It can be tricky to understand which one is the one to use. Here is how to be sure.

- If you are picking these things to be assigned different or identical properties?

If the answer is different, then you know to use permutations because the items we are giving **matter**. If the answer is identical, we have to use combinations because we just want the certain $k$ number of people to receive the item. We don't care if Person $A$ gets it before Person $B$ because the items are the same.

## 6 Circle Problems

### 6.1 Rotations

When we are dealing with let's just say the number of ways to put 7 people around a circular table, we wouldn't use the 7! **if rotations are indistinct**. This basically means if you order 4 people around a table, and rotate it 90°, those two will be counted as the same arrangement. So for these types of problems, **when rotations are considered indistinct**, the number of ways to order $n$ people around a circular table is $(n-1)!$. This is because we can rotate the table in $n$ number of ways with each person being the "starting" person. So because of this, we have to divide $n!$ by $n$ so we get rid of those cases and don't over count.

### 6.2 Reflections + Rotations

When we deal with rotations we know the total number of ways to order the people is $(n-1)!$. But now when reflections come into the picture, we have to divide by 2. Reason for this is when we reflect, we get two different ways to order them. Think about it like flipping a key chain. When you reflect it, the order seems different but it is just a different type of arrangement. So when we add rotations into the picture, we get $\frac{(n-1)!}{2}$.

**Problem 6.1**

How many different ways are there to arrange 5 keys on a keychain if rotations and reflections are not considered distinct?

*Solution.* Notice that we have to worry about reflections and rotations because it asks us for different ways. So first we have to find the number of ways through rotation only which is $(5-1)!$ or $4! = 24$. Now we can flip the keychain; therefore, we have to divide by 2 so we get rid of those overcounting. So we get $\frac{24}{2}$ or $\boxed{12.}$ □

## §7 Stars and Bars

After learning Combinations, Stars and Bars is a very useful technique. This basically allows us to figure out the number of ways we can put $n$ number of things into $k$ number of categories.

**Example 7.1**

How many ways can we put 7 cubes in 3 boxes if each box must have at least 1 cube.

This is a very classic problem using stars and bars. The formula to solving these types of problems are:

$$\binom{n-kx+k-1}{k-1}$$

In this case, $n$ number of things you want to distribute. $k$ is the number of categories you want to split up into. $x$ is the number of things you want in each box. In this case it will be 1 because you need at least 1 cube in each box. So plugging into the formula we can see that the problem simplifies to:

$$\binom{7-3+3-1}{3-1} = \binom{6}{2} = \boxed{20}$$

This formula can also be referred to as "sticks and stones," "balls and urns," "chopsticks and dumplings," etc.

*Proof.* The proof for Stars and Bars (the way I am proving it) is very visual.
Let there be $n$ number of things and $k$ number of boxes. We want to be able to distribute it into those $k$ boxes, and find the number of ways to do that. So lets start of with 2 cases. One where each box does not have to have a required number of objects, and then one where it does.

**No requirements:**
For this proof we can just let there be dividers. Those dividers can come into where ever we want to put the objects. So if we want to have 3 in one box, 2 in another box, and 4 in the last box, the dividers would look like this:

$$xxx \mid xx \mid xxxx$$

So we notice that we have exactly $k-1$ dividers. And notice that we only want to pick out how we want to choose the dividers into the $n$ objects. So all we are doing is $\binom{(n)+(k-1)}{(k-1)}$. We are just choosing $k-1$ dividers from the overall group of $n+k-1$.

**With a number of $x$ requirements in each box:**
So the proof for this is almost the same except we have a requirement. If we use the same example from above that we have 3 in one box, 2 in another box, and 4 in the last box, and say that we have a requirement of 1 in each box, we are essentially just remove 3 of the $n$ objects, and then using the regular stars and bars formula, and then adding the 3 objects again to their respective boxes. And that is exactly what the formula is doing. When we subtract $kx$ we are basically removing the requirement, then using the regular stars and bars formula to find out the number of ways to reorder the remaining objects, and then we are just adding the $kx$ back into the picture! □

**Problem 7.2** (Brilliant)
Find the number of non-negative integer solutions to

$$a+b+c+d+e+f=23$$

*Solution.* We have 6 variables that we have to use. Notice the problem says **non-negative**. This means that the value of each variable can be 0, so we don't need to assign each variable with at least 1. Now using our formula:

$$\binom{23+6-1}{6-1}=\boxed{98280}$$

□

**Problem 7.3**
Find the number of positive integer solutions to

$$x+y+z=21$$

*Solution.* Notice that this problem says **positive** integer solutions. This means that each value must have at least 1 "star" in the "box." This means that the problem is saying there are 21 objects and 3 boxes. You want to distribute them somehow such that each box has at least 1 object. Now we use the formula:

$$\binom{21-3\times 1+3-1}{3-1}=\binom{20}{2}=\boxed{190}$$

□

## 8 Pascal's Triangle

This is Pascal's triangle:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\begin{array}{ccccccccccc} & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \iddots & & & & & \vdots & & & & & \ddots \end{array}$$

Row 0 has just 1 number in it: 1. Every row after that stars with 1 and ends with 1 and all the middle terms is the sum of the two terms directly above it. For example. The 2 in Row 2, is equal to $1+1$. One cool feature about the Pascal's Triangle is that it can be rewritten to be written like this:

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ & \iddots & & & & \vdots & & & & \ddots & \end{array}$$

Notice from this, we can come up with a formula that states this:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

The reason for this is if we compare each term from the second pascal's triangle that we wrote and the first one, the terms match. The row number stays the same in the Right Hand Side and the the term number in the row gets added by 1. From this we can see that the sum of those will be the same term number as the second term in the Right Hand Side, and the next row. From these conditions, we can see that the condition applies. This is called Pascal's Identity or Pascal's Rule.

Another thing that we see is that in Row $n$, starting with Row 0, the sum of all terms in $2^n$. Looking at the second Pascal's Triangle, we see this:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

This is also known as the Combinatorial Identity

*Proof.* Let there be a polynomial $(1+x)^n$. Using the binomial theorem, we see this:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1} x \cdots + \binom{n}{n} x^n$$

Plugging in $x = 1$, we see this:

$$2^n = \binom{n}{0} + \binom{n}{1} \cdots + \binom{n}{n}$$

□

**Problem 8.1** (1992 AIME Problem 4)

In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$?

*Solution.* Let $n$ be the row number, and $k$ be one of the numbers in that row. So we know the ratio is $3:4:5$. This means that $\binom{n}{k-1}:\binom{n}{k}:\binom{n}{k+1}=3:4:5$. This means that we can write a simple setup that looks like:

$$\frac{\binom{n}{k-1}}{3}=\frac{\binom{n}{k}}{4}=\frac{\binom{n}{k+1}}{5}.$$

Instead of looking at all three, we can split them up. We can look at the first 2:

$$\frac{\binom{n}{k-1}}{3}=\frac{\binom{n}{k}}{4}$$

Simplifying this we can see:

$$\frac{n!}{3(k-1)!(n-k+1)!}=\frac{n!}{4k!(n-k)!}$$

Looking at $(n-k)!$ and $(n-k+1)!$ we can cross out $(n-k+1)!$ and $(n-k)!$ to get $(n-k+1)$ left over on one side. Similarly we can do the same for $k!$ and $(k+1)!$. So simplifying the above equation we see:

$$\implies \frac{1}{3(n-k+1)}=\frac{1}{4k} \implies n-k+1=\frac{4k}{3}$$

Solving for $k$ in terms of $n$ (Since we want to solve for $n$), we get:

$$k=\frac{3n+3}{7}$$

Now we can simplify the second and third term in the first equation that we got:

$$\frac{n!}{4k!(n-k)!}=\frac{n!}{5(k+1)!(n-k-1)!}$$

$$\implies \frac{1}{4(n-k)}=\frac{1}{5(k+1)}$$

$$\implies 4n-4k=5k+5 \implies 9k=4n-5$$

Substituting the value of $k$ into this equation we get:

$$9\left(\frac{3n+3}{7}\right)=4n-5 \implies \frac{27n+27}{7}=4n-5$$

Now we can easily solve for $n$ and we get $n=\boxed{62}$ □

## §9 Exercises

**Exercise 9.1** (2019 AMC8)**.** Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

**Exercise 9.2.** Find the number of words, with or without meaning, that can be formed with the letters of the word "SWIMMING"?

**Exercise 9.3** (Stack Exchange)**.** How many distinct permutations of the string "NADAMADRID" have the word "DAM" appearing in them?

**Exercise 9.4.** If there are 9 points in a plane, and no 3 of the points lie along the same line, how many unique lines can be drawn between pairs of these 8 points?

**Exercise 9.5.** How many triplets of non-negative integers are there such that

$$a + b + c = 20$$

**Exercise 9.6.** What is the sum of all of the terms in Row 10 in Pascal's Triangle?

**Exercise 9.7** (2017 AMC10)**.** Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

**Exercise 9.8** (Brilliant)**.** How many ways are there to choose a 5-letter word from the 26-letter English alphabet with replacement, where words that are anagrams are considered the same?