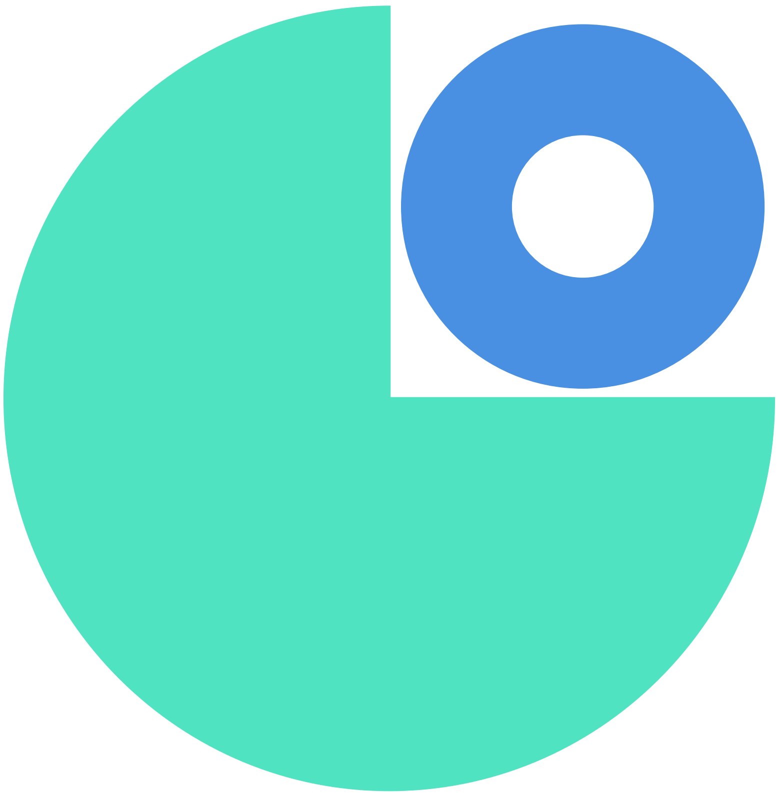
The CALT
Basic Geometry

WIZ



Diagrams are meaningful drawings...

Contents

§1 Introduction

§1.1 Why Bother?

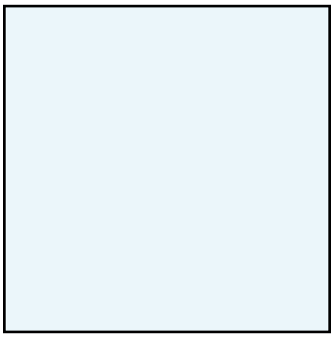
Now, why should we bother to find the area or the perimeter of a random figure which barely has any meaning to us. Well, there is one reason for it, you are made up from shapes, you live in shapes, you draw with shapes, you write with shapes and...
So, to conclude, shapes are everywhere, and that is why we have to know how to measure them.
So, lets do a quick overview of how to calculate the area and perimeter of basic shapes and then we will move on to a few funky ones.

§1.2 Quadrilaterals

Okay, so what are quadrilaterals? Well, quadrilaterals are figures with **4 sides**. Here are a few basic quadrilaterals that you ought to know:

§1.2.1 Square

Here is a square:



Lets look over a few properties of a square:

1. All four sides of a square are same length, they are equal:
2. Opposite side of a square are parallel
3. All four angles of a square are right angles:
4. Sum of the angles of a square are equal to 360 degrees:
5. Diagonal of a square are same length:
6. Each diagonal of a square divides its into two equal symmetrical area

Formula

Area of a Square: s^2

Perimeter of a Square: $4s$

§1.2.2 Rectangle

Here is a rectangle:

Lets look over a few properties of a rectangle:

1. A rectangle is a quadrilateral
2. The opposite sides are parallel and equal to each other
3. Each interior angle is equal to 90 degrees
4. The diagonals bisect each other
5. Both the diagonals have the same length

Formula

Area of a Rectangle: lb

Perimeter of a Rectangle: $2(l + b)$

§1.2.3 Parallelogram

Here is a parallelogram:

Lets look over a few properties of a parallelogram:

1. The opposite sides are congruent.
2. The opposite angles are congruent.
3. The consecutive angles are supplementary.

4. If anyone of the angles is a right angle, then all the other angles will be the right angle.
5. The two diagonals bisect each other.
6. Each diagonal bisects the parallelogram into two congruent triangles.

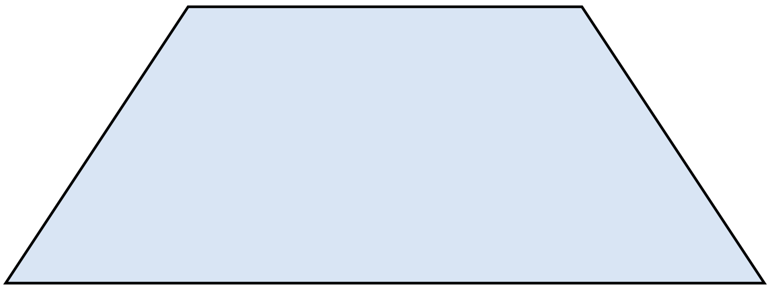
Formula

Area of a Parallelogram: bh

Perimeter of a Parallelogram: $2(a+b)$

§1.2.4 Trapezoid

Here is a trapezoid:



Lets look over a few properties of a Trapezoid:

1. The properties of trapezoid apply by definition (parallel bases).
2. The legs are congruent by definition.
3. The lower base angles are congruent.
4. The upper base angles are congruent.
5. Any lower base angle is supplementary to any upper base angle.
6. The diagonals are congruent.

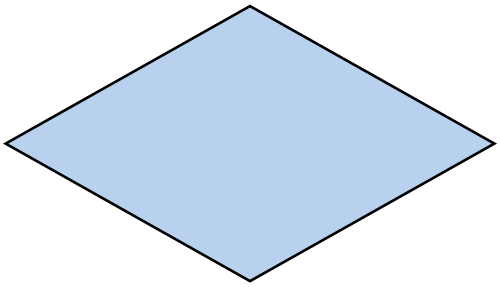
Formula

Area of a Trapezoid: $\frac{b_1 + b_2}{2} \times h$

Perimeter of a Trapezoid: $b_1 + b_2 + m + n$

§1.2.5 Kite

Here is a Kite:



Lets look over a few properties of a Kite:

- Two pairs of sides known as consecutive sides are equal in length.
- One pair of diagonally opposite angles is equal in measurement. These angles are said to be congruent with each other.
- The diagonals meet each other at 90°, this means that they form a perpendicular bisection.

Formula

Area of a Kite: $\frac{pq}{2}$ where p and q are diagonals.

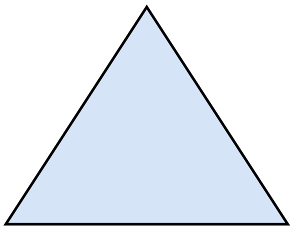
Perimeter of a Kite: $2a + 2b$

§1.3 Triangles

Triangles are figures which have 3 sides. They can be classified based on angles and lengths of sides. Lets look at a few triangles:

§1.3.1 Equilateral Triangle

Here is an equilateral Triangle:



Lets look over a few properties of an Equilateral Triangle:

- All sides are of equal length.
- All medians are of equal length

- All interior angles are 60°
- It is a regular polygon

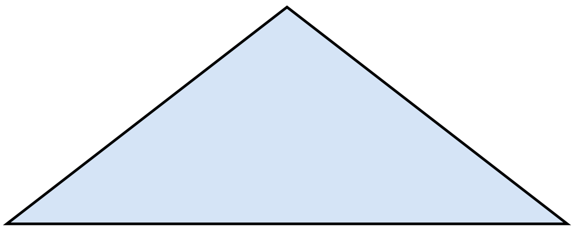
Formula

Area of an Equilateral Triangle: $\frac{\sqrt{3}}{4}s^2$

Perimeter of an Equilateral Triangle: $3s$

§1.3.2 Isosceles Triangle

Here is an isosceles triangle:

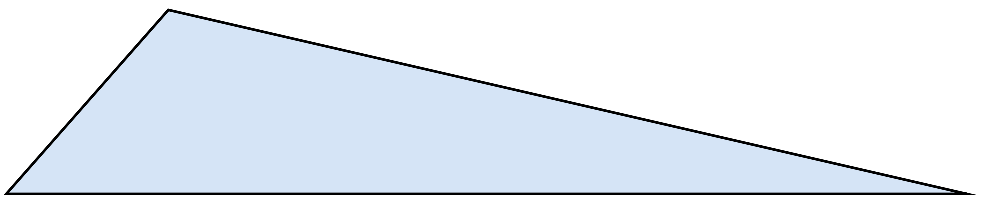


Lets look over a few properties of an Isosceles Triangle:

- Two sides are congruent to each other.
- The third side of an isosceles triangle which is unequal to the other two sides is called the base of the isosceles triangle.
- The two angles opposite to the equal sides are congruent to each other. That means it has two congruent base angles and this is called an isosceles triangle base angle theorem.
- The angle which is not congruent to the two congruent base angles is called an apex angle.
- The altitude from the apex of an isosceles triangle bisects the base into two equal parts and also bisects its apex angle into two equal angles.
- The altitude from the apex of an isosceles triangle divides the triangle into two congruent right-angled triangles.

§1.3.3 Scalene Triangle

Here is a scalene triangle:

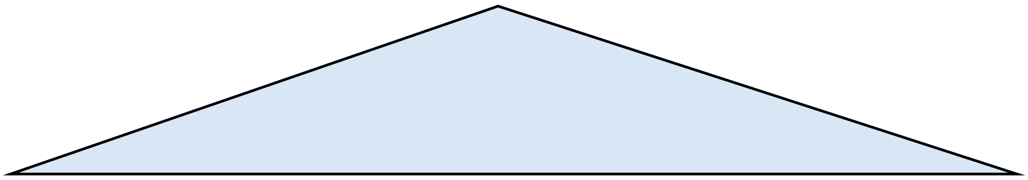


Lets look over a few properties of a Scalene Triangle:

- It has no equal sides.
- It has no equal angles.
- It has no line of symmetry.
- It has no point symmetry.
- The angles inside this triangle can be an acute, obtuse or right angle.

§1.3.4 Obtuse Triangles

Here is an obtuse triangle:

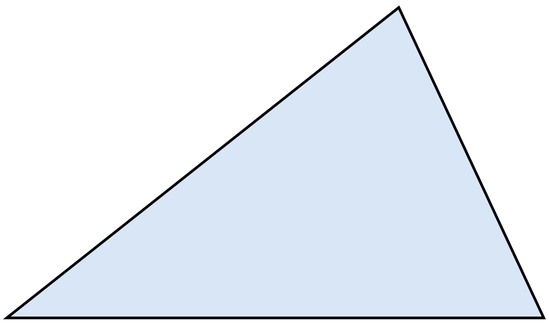


Lets look over a few properties of an Obtuse Triangle:

- The longest side of an obtuse triangle is the one opposite the obtuse angle vertex.
- An obtuse triangle may be either isosceles (two equal sides and two equal angles)

§1.3.5 Acute Triangles

Here is an acute triangle:

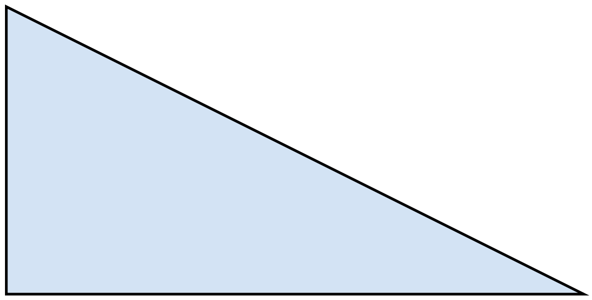


Lets look over a few properties of an Acute Triangle:

- All equilateral triangles are acute triangles.
- Any triangle in which the Euler line is parallel to one side is an acute triangle.
- Acute triangles can be isosceles, equilateral, or scalene.
- The longest side of an acute triangle is opposite the largest angle.

§1.3.6 Right Angle Triangles

Here is a right angle triangle:



Lets look over a few properties of a Right Angle Triangle:

- The sum of all three interior angles is 180°.
- The larger interior angle is the one included by the two legs, which is 90°.
- The sum of the two smaller interior angles is 90°
- Uses the Pythagorean Theorem which states that $a^2 + b^2 = c^2$

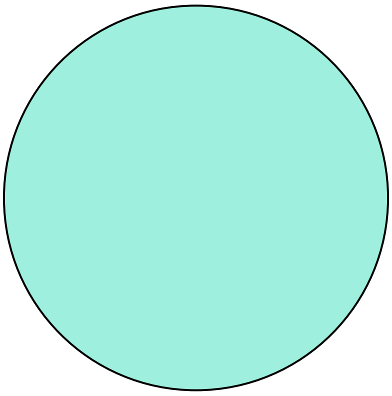
Formula

Area of a Right Angle Triangle: $\frac{ab}{2}$

Perimeter of a Right Angle Triangle: $a + b + c$

§1.4 Circular Figures

Now, lets come to the circles. Here is a circle:



Lets look over a few properties of a Circle:

- The circles are said to be congruent if they have equal radii
- The diameter of a circle is the longest chord of a circle
- Equal chords and equal circles have the equal circumference
- The radius drawn perpendicular to the chord bisects the chord
- Circles having different radius are similar
- A circle can circumscribe a rectangle, trapezium, triangle, square, kite
- A circle can be inscribed inside a square, triangle and kite
- The chords that are equidistant from the centre are equal in length
- The distance from the centre of the circle to the longest chord (diameter) is zero
- The perpendicular distance from the centre of the circle decreases when the length of the chord increases
- If the tangents are drawn at the end of the diameter, they are parallel to each other
- An isosceles triangle is formed when the radii joining the ends of a chord to the centre of a circle

Now a few of these terms might not be well known at all. Don't worry, because we will cover them later. After that you can come back and overview some of this.

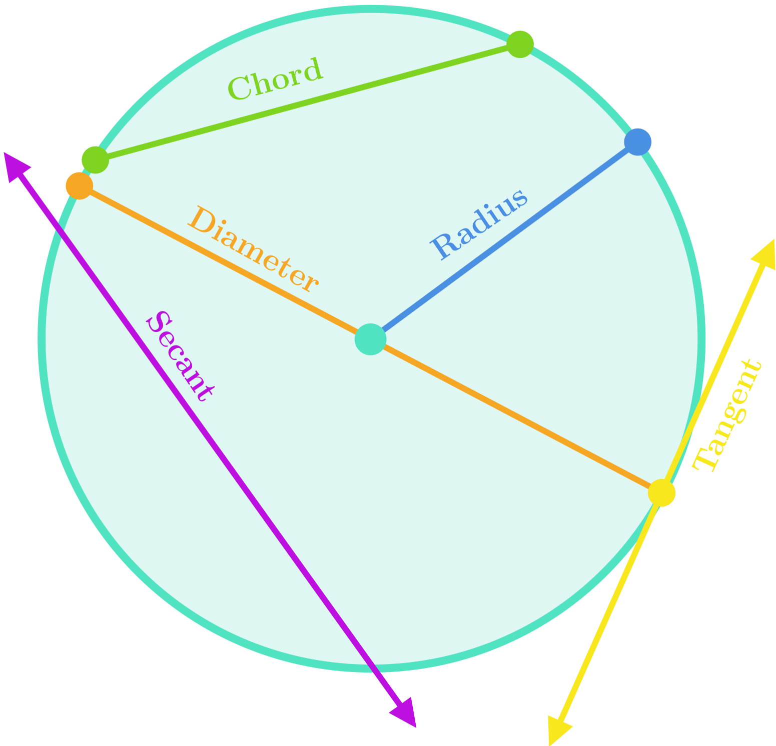
Formula

Area of a Circle: πr^2

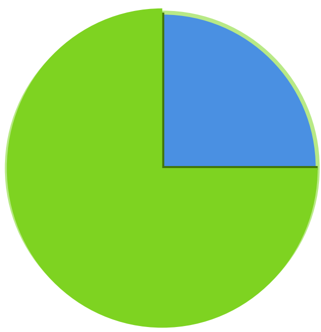
Perimeter/Circle of a Circle: πr or πd

where d is the diameter, r is the radius and π is 22/7 or 3.1415...

Now, back to that pesky terminology. Here's a simple and brief visual representation of them.



Now, apart from that, we know that a sector is part of a circle. Like a slice of pizza is like a sector of the whole thing. The blue part here is a sector of its circle.



Lets look at how to calculate the area of a sector. A sectors central angle will have a degree measure of θ. Now, the whole circle has an area of 360° and we want a fraction of that. This is when we come to the formula.

Formula

Area of a Sector of a circle: $\frac{\theta}{360} \times \pi r^2$

where θ is the central angle, r is the radius and π is 22/7 or 3.1415...

§1.5 Polygons

Now, what is a polygon?
A polygon is a two-dimensional figure that has a finite number of sides. The sides of a polygon are made of line segments connected to each other end to end. The line segments of a polygon are called sides. The point where two line segments meet is called vertex or corners. An angle is formed at each of these corners.

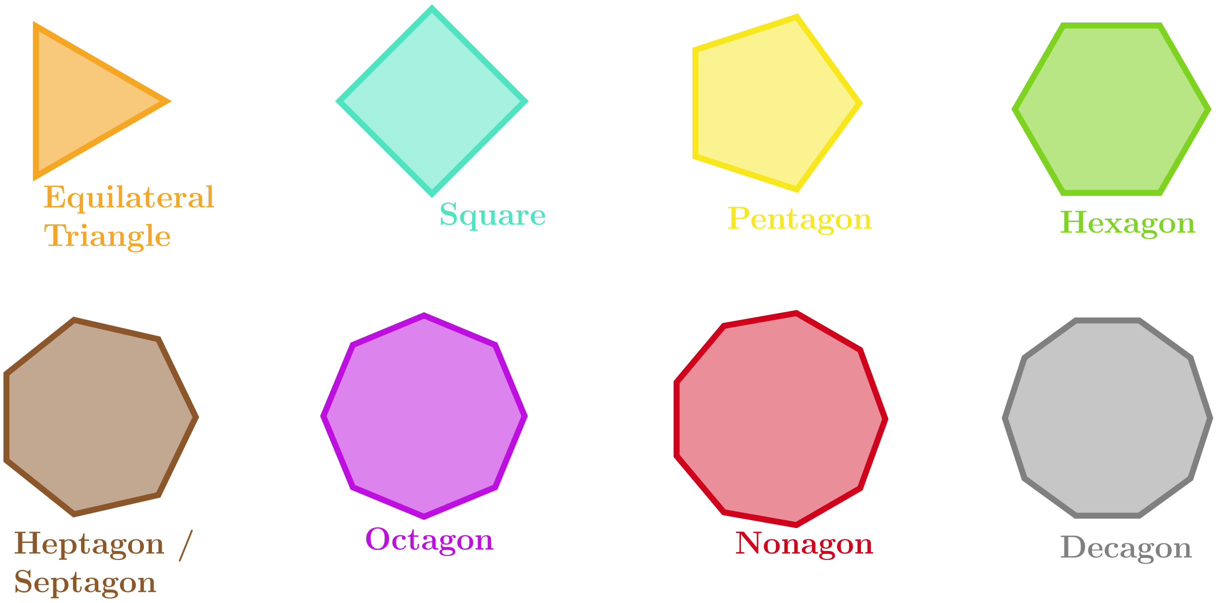
Is circle a polygon? Take a look at the definition again and see if it follows all the criteria.

All quadrilaterals and triangles which we explored above are polygon

There are 4 types of Polygons:

Regular Polygons

If all the sides and interior angles of the polygon are equal, then it is known as a regular polygon. Here are a few Regular polygons with their names:



Irregular Polygons

If all the sides and the interior angles of the polygon are of different measure, then it is known as an irregular polygon.

Convex Polygons

If all the interior angles of a polygon are strictly less than 180 degrees, then it is known as a convex polygon. The vertex will point outwards from the centre of the shape.

Concave Polygons

If one or more interior angles of a polygon are more than 180 degrees, then it is known as a concave polygon. A concave polygon can have at least four sides. The vertex points towards the inside of the polygon.

§2 Pythagorean Theorem

§2.1 Brief History and Derivation

Most of the people believe that the theorem has long been associated with Greek mathematician Pythagoras, the truth is, that it is actually far older. Four Babylonian tablets indicate some knowledge of the theorem, with a very accurate calculation of the square root of 2 (the length of the hypotenuse of a right triangle with the length of both catheti equal to 1) and lists of special integers known as Pythagorean triples. Nevertheless, the theorem came to be credited to Pythagoras.

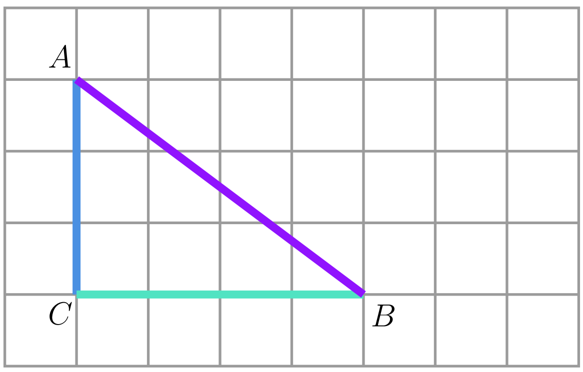
§2.2 The Theorem

The Pythagorean Theorem might just be one of the most vastly used and most important theorem not only in geometry, but it is applied almost everywhere. Here is the Pythagorean Theorem:

Theorem
For any right angle triangle with legs a and b and hypotenuse c, we have that

$$a^2 + b^2 = c^2.$$

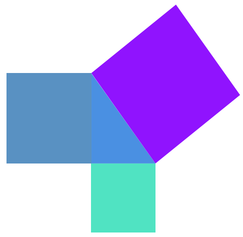
Now, lets try it for ourselves. Lets see if it really works. Lets take out some grid paper or graph paper. Make a line with length of 3 grids vertically and make another linewith length of 4 grids horizontally, sharing one of the endpoints with the other line. Now mark the common endpoint as C, and the 2 other points as A and B. Now, draw line $\overline{AB}$ and measure its length with a ruler. Did you get 5 grids as the answer? Well, then try putting the legs as 3 and 4 and the hypotenuse as 5 in the formula above. Does this work? So is this true? Your diagram should look somewhat like this.



Well, we see that indeed $3^2 + 4^2 = 5^2$ since $9 + 16 = 25$. This is where we have come across our first Pythagorean Triplet.

A Pythagorean Triplet is a triplet in the form of (a, b, c) such that it follows the pythagorean theorem which states that $a^2 + b^2 = c^2$. So, our first Pythagorean Triplet is $(3, 4, 5)$.

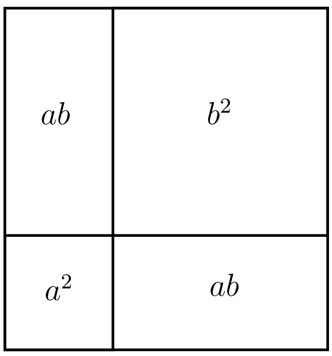
Now what we just did was a visual proof. It does not show how the Pythagorean Theorem works in all cases. So we have to work on proving that. See the image below. That is also a visual proof. You can prove it using a water tank or many such things.



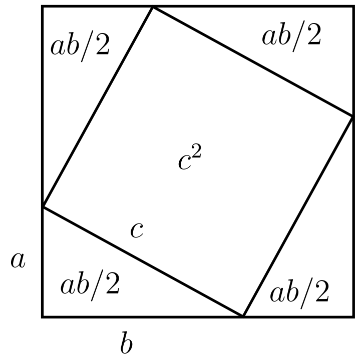
§2.3 Proving the Theorem

Now lets come to proving the theorem. This handout will only contain one way out of the many extensive ways of proving it but we will leave that to you.

So, first lets start with a sqaure with side $a + b$. We know that $(a + b)^2 = a^2 + 2ab + b^2$ using expansion. So, lets start by drawing it with marked areas.



Now we notice that we can draw it another way as well. Not using expansion, but through drawing triangles and a giant square in the centre. Doing so and marking the area of each region we get



Now we notice that both the squares had the same side length so they must have the same area. With this observation we get

$$a^2 + b^2 + 2ab = c^2 + 4\left(\frac{ab}{2}\right)$$
$$a^2 + b^2 + 2ab = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

Boom! And it is proved. There are multiple other methods that you can use to prove this theorem, each fascinating in its very own way.

Note. We could have also used Al - Kashi's Law of Cosines to prove this theorem.

§2.4 Pythagorean Triplets

A Pythagorean Triplet is a triplet in the form of (a, b, c) such that it follows the pythagorean theorem which states that $a^2 + b^2 = c^2$.

Here are a few must know Pythagorean Triplets:

3 - 4 - 5	**5 - 12 - 13**	**8 - 15 - 17**
20 - 21 - 29	**12 - 35 - 37**	**9 - 40 - 41**
28 - 45 - 53	**11 - 60 - 61**	**48 - 55 - 73**

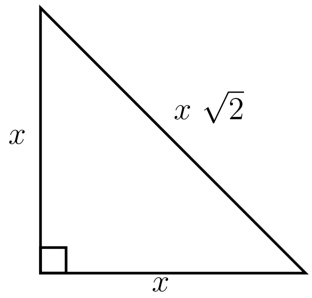
Can you confirm if all of these are correct my plugging in values?

§2.5 Special Triangles

Lets see how the Pythagorean theorem helps us find special triangles. So far, there are 2 basic special right angle triangles. Here are both of them and how the Pythagorean theorem helps us find the side ratios briefly.

§2.5.1 45 - 45 - 90

This kind of triangle is also known as an isosceles right angle triangle. Lets take a look at it with leg side as x.



Now, since this is a right angle triangle, we can solve for the hypotenuse using the Pythagorean Theorem. Lets keep the hypotenuse as h and solve. We get

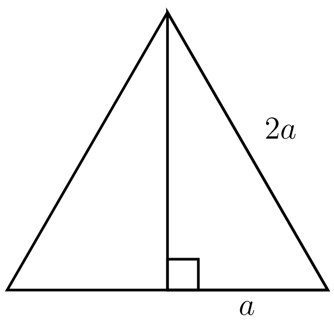
$$\begin{aligned} x^2 + x^2 &= h^2 \\ 2x^2 &= h^2 \\ h &= x\sqrt{2} \end{aligned}$$

So we have successfully found it.

Do you think this triangle can be formed using any of the geometric figures we learnt above?

§2.5.2 30 - 60 - 90

Lets start with an equilateral triangle. We can draw the altitude. Lets say the side length is $2a$.



We can see that since 2 of the angles are 60° and 90°, the third one must be 30° since angles of a triangle add up to 180°. Since this is a right angle triangle, we can find the altitude using the Pythagorean theorem. Lets keep the altitude as h.Using the Pythagorean theorem, we get

$$\begin{aligned} a^2 + h^2 &= (2a)^2 \\ a^2 + h^2 &= 4a^2 \\ h^2 &= 3a^2 \\ h &= a\sqrt{3} \end{aligned}$$

So we have successfully found the ratio of the sides. Remember how we found it, using an equilateral triangle and then dropping an altitude.

§2.6 Problems

Example

Find the area of a right triangle (in inches square) with length of hypotenuse as 13 inches and if the length of one of the catheti is 1 foot.

Solution. We can use the Pythagorean theorem and keep the unknown value as x and solve for it. Notice that 1 foot = 12 inches. So we get $x^2 + 12^2 = 13^2$. Simplifying we get $x^2 = 25$ or $x = 5$. So the area is $\frac{ab}{2} = \frac{5 \times 12}{2} = \boxed{30}$.

Example

Anna drives 7 miles north from her home. She then turns right and drives 24 miles further. What is the shortest distance in miles from her home to where she is now?

Solution. We see that she has gone 7 miles north and 24 miles east. So through the pythagorean theorem, we get the distance to be $\sqrt{7^2 + 24^2} = \sqrt{625} = \boxed{25}$.

Example

Find the area of a triangle with side length of 6.

Solution. We know the altitude of the triangle should be $\frac{6}{2} \times \sqrt{3} = 3\sqrt{3}$. Now we can simply find the area using the formula $A = b \times h \times \frac{1}{2} = 3\sqrt{3} \times 6 \times \frac{1}{2} = \boxed{9\sqrt{3}}$

Note. Can you figure out a generalized way to find the area of an equilateral triangle with side length s?

Example

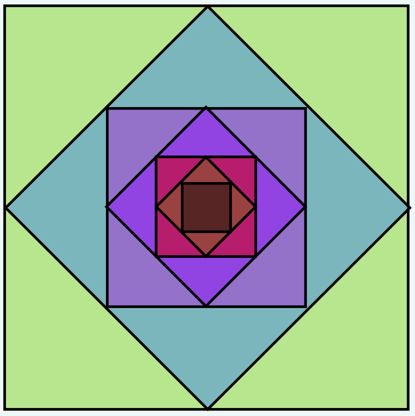
Find the value of $\frac{11^2 + 60^2}{61^2}$.

Solution. Now, this might look like an algebra problem, but we can use the knowledge through the Pythagorean theorem that $11^2 + 60^2 = 61^2$. This means that since both the numerator and denominator are of equal value, the answer will be $\boxed{1}$.

§3 A few Examples

Example

Find the ratio between the area of the smallest square to the largest square in the following figure.

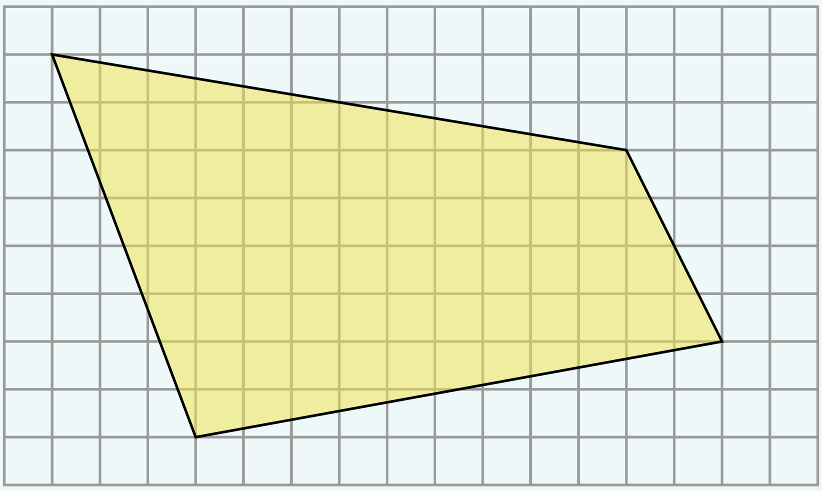


Solution. Lets say the largest square has a length of $2x$. So the area of this square will be $4x^2$. The turquoise square will have area $2x^2$. The the light purple square will have area x^2. Then the dark purple square will have area $x^2/2$. Then the red square will have area $x^2/4$. Then the light brown square will have area $x^2/8$. Then, finally, the area of the smallest square will be $x^2/16$. So the ratio will be

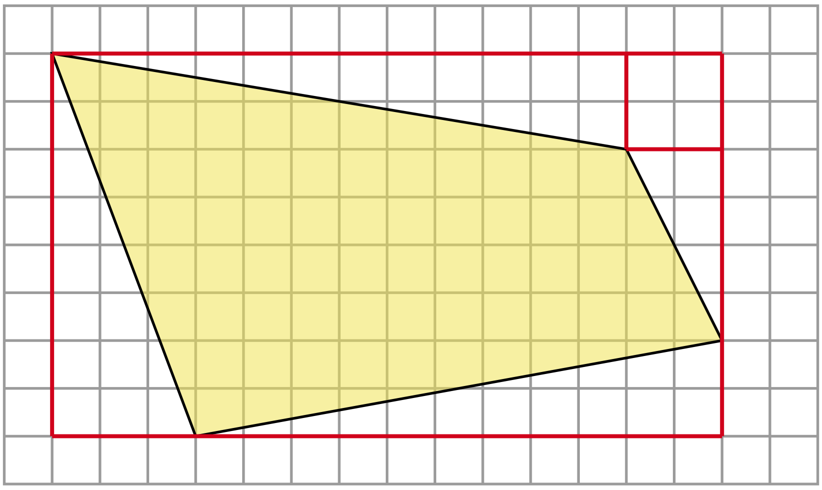
$$\frac{x^2/16}{4x^2} = \frac{x^2}{64x^2} = \boxed{1:64}$$

Example

Find the area of the following figure if each square has dimension of 1×1.



Solution. We can begin by adding a few extra lines.

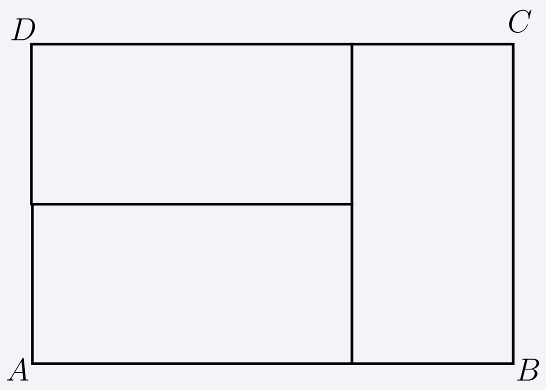


The area of the total rectangle is $8 \times 14 = 112$.
The area of the triangles will be $12 + 12 + 11 = 35$.
The area of the square is $2 \times 2 = 4$.
So the area of the shaded area will be $112 - 35 - 4 = \boxed{73}$.

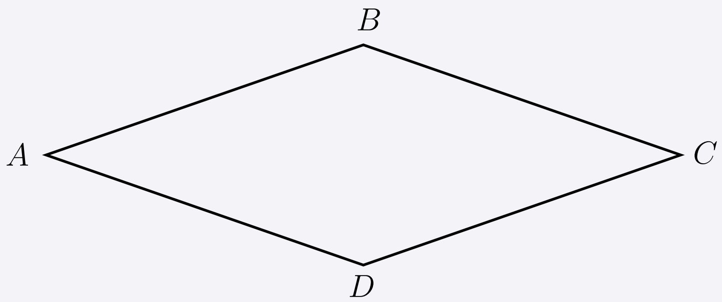
Note. We could have also used Pick's Theorem.

§4 Exercises

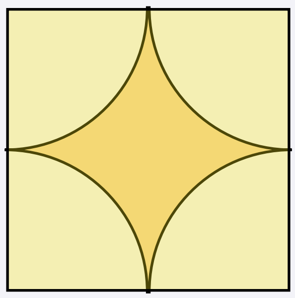
Problem 1 (2019 AMC 8)
Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



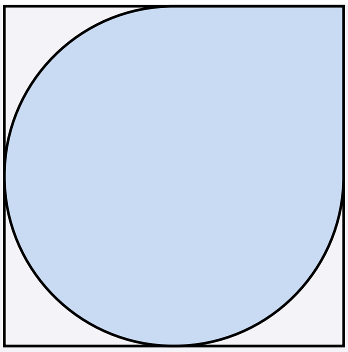
Problem 2 (2019 AMC 8)
Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$?



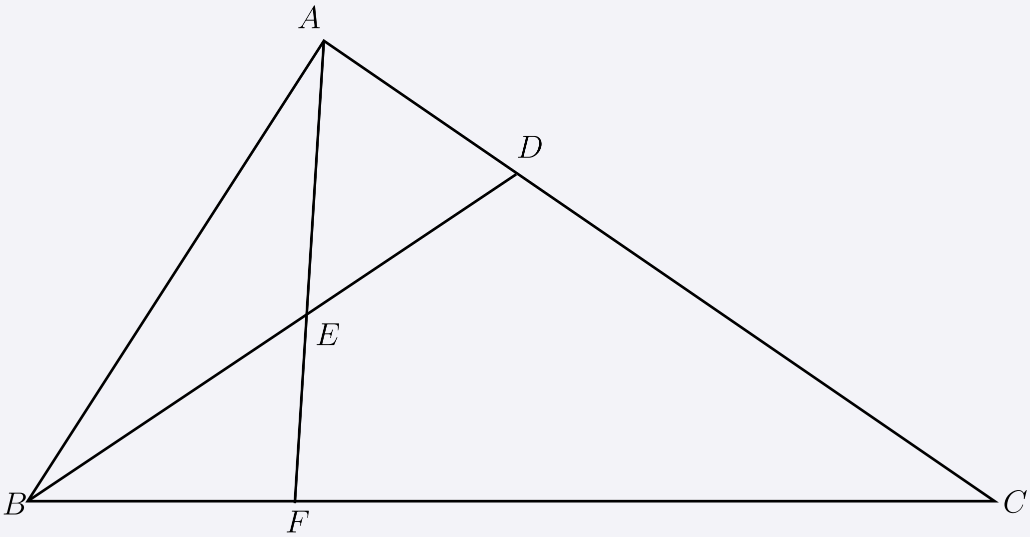
Problem 3
Find the area of the dark orange region if the side of the square is 28.Use $\pi = \frac{22}{7}$.



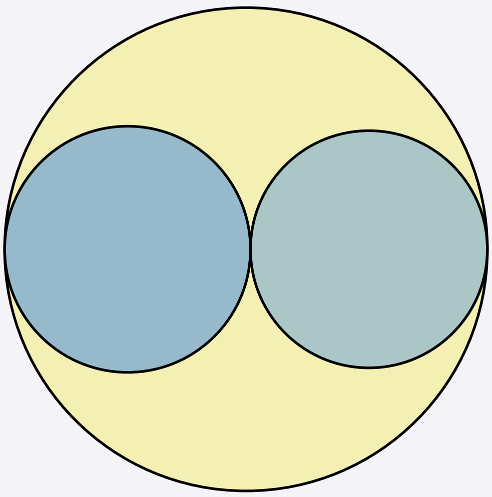
Problem 4
Find the area of the shaded region if the square has side 14. Use $\pi = \frac{22}{7}$.



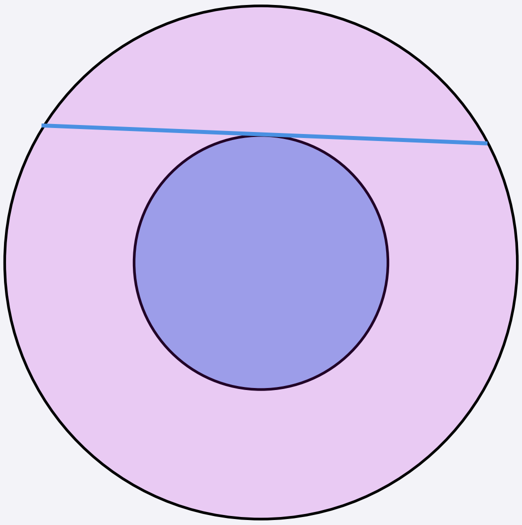
Problem 5 (2019 AMC 8)
In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



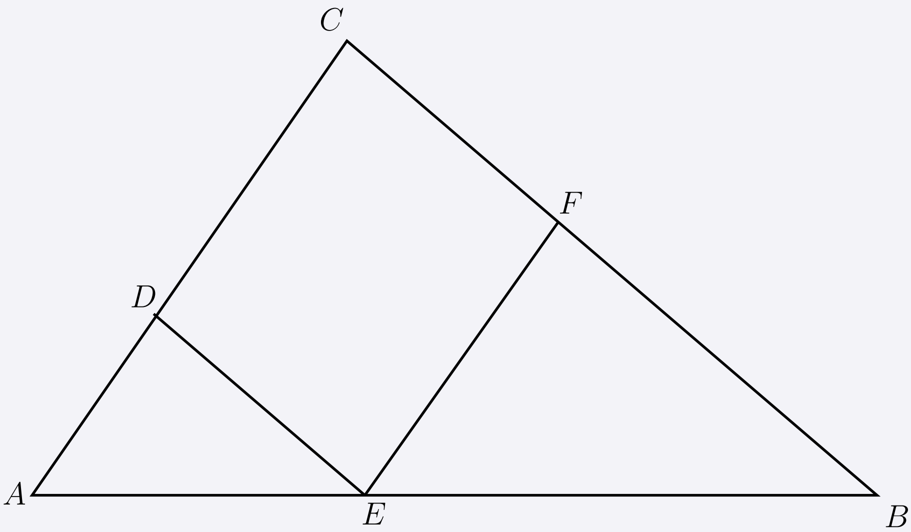
Problem 6 (2018 AMC 8)
In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the yellow region, in square units?



Problem 7
Find the difference between the area of the 2 circles, if the chord has length 70. Use $\pi = \frac{22}{7}$.

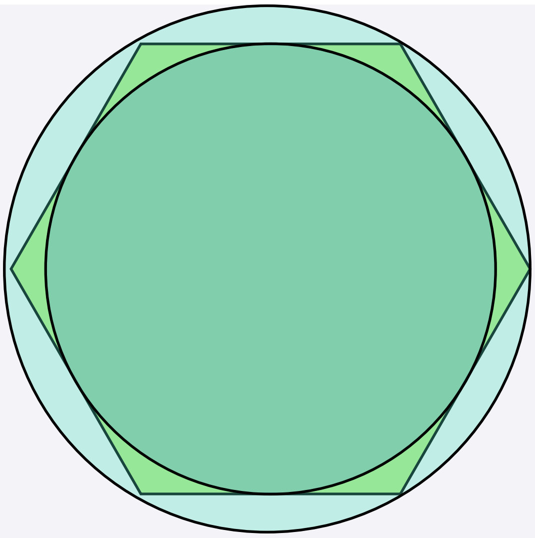


Problem 8 (2018 AMC 8)
In $\triangle ABC$, a point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$?

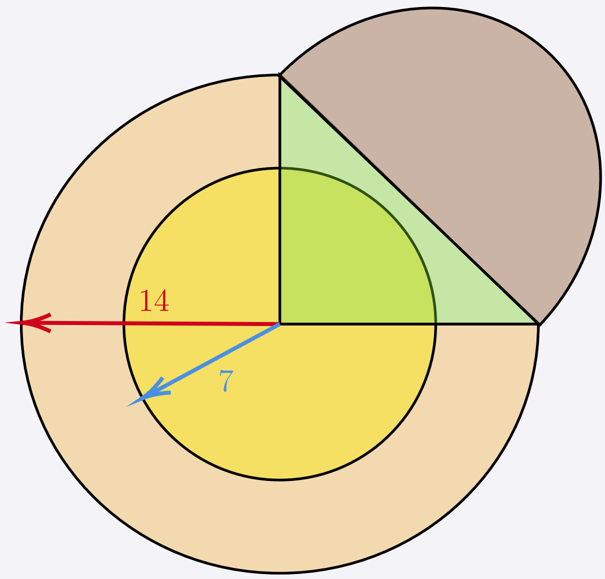


And here comes a completely different type...

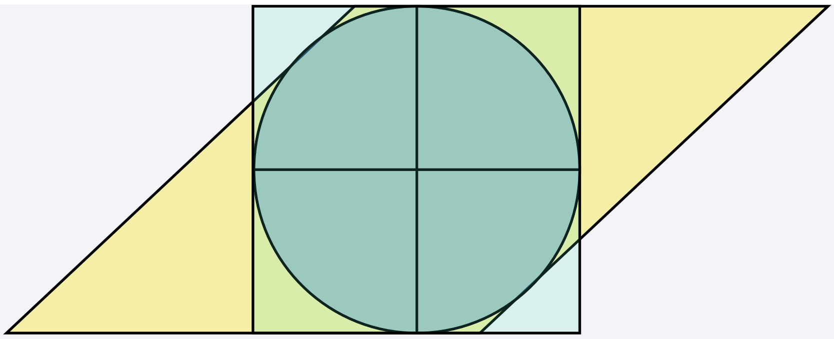
Problem 9 (Just the right amount of bashiness)
Find the ratio of the area of the small circle to the large circle, if the length of the side of the regular hexagon is 14. Use $\pi = \frac{22}{7}$.



Problem 10 (Misconception by many mathletes)
A new donut has come out in the local cafe. Below is the diagram. The donut costs \$714. Find the cost per unit area of the donut. Use $\pi = \frac{22}{7}$.



Problem 11 (Extreme Troll Problem, again misconception)
A circle is inscribed in a square and a parallelogram. The square has side 14 and the parallelogram has base length of 21 and 2 side lengths of $14\sqrt{2}$. Find the ratio between the area of the circle to the area of the parallelogram to the area of the square. Use $\pi = \frac{22}{7}$.



Note. It might be a trick question...

Problem 12

If the dimensions of an individual rectangle in the figure is 7×10, then find the area of the shaded regions.