Basics of Trigonometry

Soham Garg and Alon Ragoler

April 2021

Contents

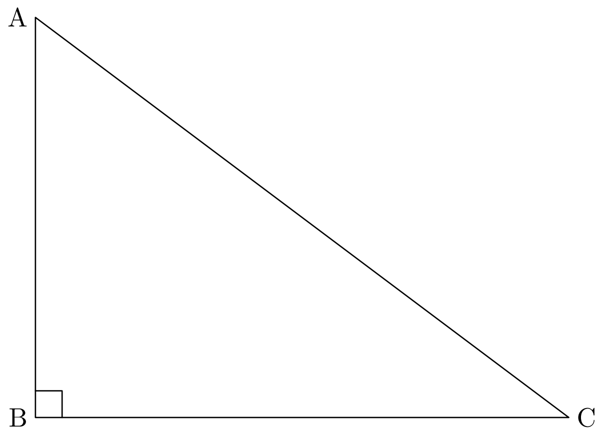
Definition 0.1 (Radians)**.** There are 2π radians in every circle, a.k.a $360°$. This is another way to measure the angle measures.

1 Introduction

In this handout, we will be going over the basics of trigonometry. We will go over the formula of sine, cosine, and tangent in right angle triangles. We will also cover two formulas named Law of Cosines and Law of Sines. Both of these formulas and these three functions are very useful in figuring out angles and side lengths of triangles.

2 Right Triangle Trigonometry

The names of the three functions that we are going to be covering in this section are called sine, cosine, and tangent. One mnemonic that is helpful is SOH CAH TOA. It basically says Sin = Opposite/Hypotenuse, Cos = Adjacent/Hypotenuse, and Tangent = Opposite/Adjacent. Let's take a look at a picture of a right angle triangle and let's label some points. Writing sin, cos, tan functions will be covered in the next handouts and will be explained in detail.



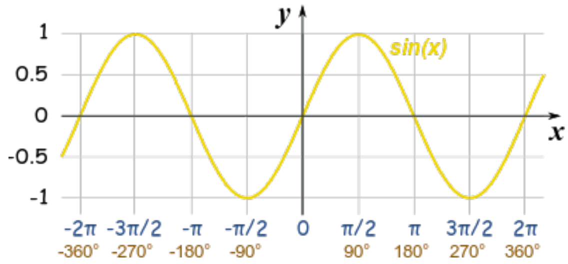
2.1 Sine

The first function that we are going to cover is sine. This is a function that allows us to figure the side length of the hypotenuse, and the opposite side length from the angle. What I mean is that in a $\triangle ABC$ with a right angle of $\angle B$, $\sin A$ is the value of $\frac{BC}{AC}$. That's why most people say Sin is opposite over hypotenuse.

This allows us to figure out the side length of the triangle if we are given the angle of A and another side length. So assume we are given that the angle measure of A is $50°$ and the side length of AC is 11, and we want to figure out the length of BC. We can set up our equation as

$$\sin 50° = \frac{BC}{11} \implies BC = 11 \times \sin 50° \approx \boxed{8.4}$$

Figure 1: Graph of a sin(x) function (Mathsisfun)



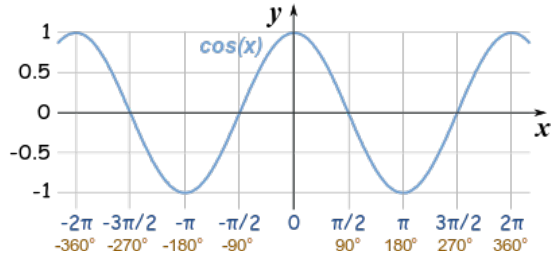
2.2 Cosine

The second function is called cosine. Like the other function, this also has a specific "formula" to it. Instead of opposite over hypotenuse, this function's formula is adjacent over hypotenuse. So what is adjacent in this case? If we look at $\cos A$ we know that AC is the hypotenuse. Because of this the adjacent side length will be AB. Now if we had $\cos B$, the answer to this is 0 because there is no "adjacent" side length in this case. Now if we look at $\cos A$ we know that AC is 11, so what is AB? We can use the Pythagorean Theorem since we already know BC, but let's try and use the cosine function. We know that the angle measure of Angle A is 50°, so plugging it into the formula we learned, we get:

$$\cos 50 = \frac{AB}{11} \implies AB = 11 \times \cos 50 \approx \boxed{7.1}$$

We can also confirm this answer by the Pythagorean Theorem!

Figure 2: Graph of a cos(x) function (Mathsisfun)



Exercise 2.1. Try and confirm this with the Pythagorean Theorem.

2.3 Tangent

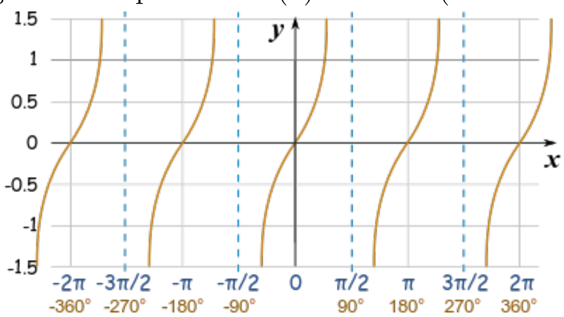
The third and last basic function is called tangent or in short form tan. The way we use the tan function is by getting the value of the side length opposite to the angle and diving by the adjacent side length. So if we want to figure out the $\tan C$ we have to solve for $\tan C = \frac{AB}{BC}$.

Since we are given that $\angle A = 50°$, we can instantly figure out that $\angle C = 40°$. Let's say we know that $BC = 8.4$. Let's try and see if we can use tangent to cross-verify what AB is. We have the following statement:

$$\tan 40° = \frac{AB}{8.4} \implies AB = \tan 40° \times 8.4 \approx \boxed{7.05}$$

.

Figure 3: Graph of a tan(x) function (Mathsisfun)



As we can see from the graphs, the dotted lines are asymptotes. This means that the function never has an x value at those values. We can see a pattern when these asymptotes occur. They only occur at values of $\frac{\pi}{2} + \pi x$ where $x \in \mathbb{Z}$.

We know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\tan\theta$ can be undefined when $\cos\theta = 0$. This happens when the point $(\cos\theta, \sin\theta)$ is on the y-axis. This means that $\tan\theta$ is undefined at 90° and at 270°. Essentially $\tan\theta$ is undefined at $90 + 180x$ degrees, where $x \in \mathbb{Z}$ which is equivalent to what we found in the paragraph above but in degrees.

Theorem 2.2 (Sine, Cosine, Tangent)

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$

2.4 Cosecant, Secant, and Cotangent

There are three more trigonometric functions in addition to sin, cos, and tan. They are fairly simple to remember, since each of them are just the reciprocal of one of the "base" trigonometric functions.

Cosecant, or csc, is defined as

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Opposite}}$$

Secant, or sec, is defined as

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Adjacent}}$$

Cotangent, or cot, is defined as

$$\cot\theta = \frac{1}{\tan} = \frac{\text{Adjacent}}{\text{Opposite}}$$

Notice that csc is actually the reciprocal of sin, NOT cos. Similarly, sec is the reciprocal of cos, NOT sin.

2.5 Inverses

Let's say that we have only the lengths of BC and AC and we want to figure out $\angle A$. We can still achieve this using the sin function! Think of the sin function as a regular function ($f(x)$). If we want to figure out $f(x) = y$ and we have the value of y, we can use the inverse of f and get $x = f^{-1}(y)$. We can do the same thing here and write

$$\sin A = \frac{8.4}{11} \implies A = \sin^{-1}\left(\frac{8.4}{11}\right)$$

You can plug this into the calculator and get your answer!

2.5.1 Restrictions

Like all functions, inverses of trigonometric functions also have restrictions on what the values can be.

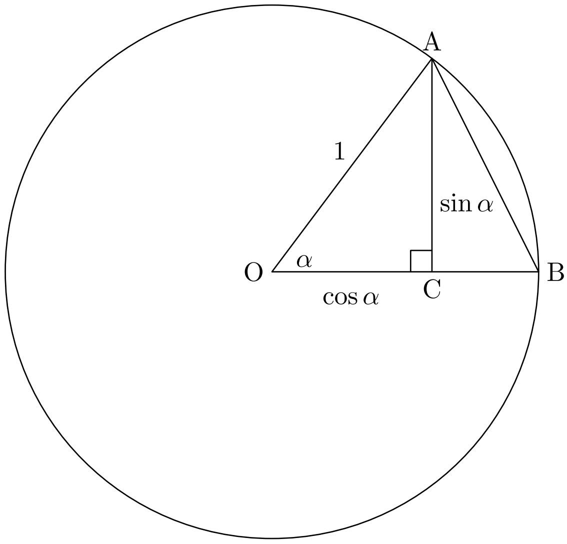
Restrictions		
Function	Domain	Range
$\sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\tan^{-1}(x)$	$\mathbb{R}$ (This just means all real numbers	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\csc^{-1}(x)$	$x \leq -1 \cup x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ where $y \neq 0$
$\sec^{-1}(x)$	$x \leq -1 \cup x \geq 1$	$0 \leq y \leq \pi$ where $y \neq \frac{\pi}{2}$
$\cot^{-1}(x)$	$\mathbb{R}$	$0 \leq y \leq \pi$

3 The Unit Circle

So far, we've only been dealing with angles that are part of right triangles, specifically the acute angles. This is extremely useful, but has its limitations. For one, we cannot generalize the trigonometric functions to negative numbers and numbers greater than 90. To do so, we will invoke the Unit Circle definition.

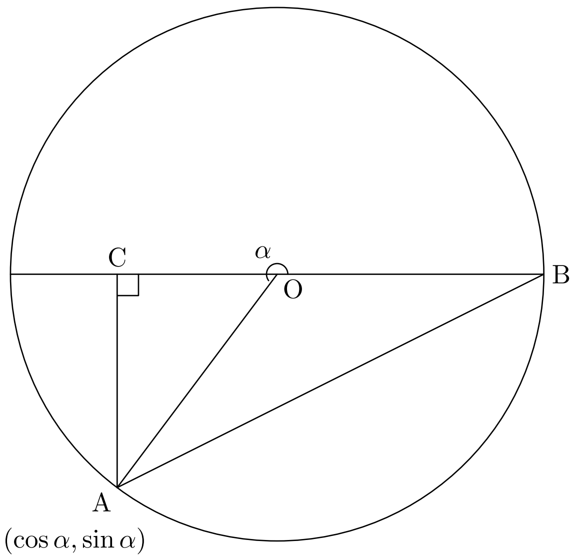
3.1 Unit Circle Definition

Consider a circle centered at $O(0, 0)$ with radius 1 (this is called the Unit Circle). Let $B(1, 0)$ and A be a point in the first quadrant such that $\angle AOB = \alpha$, for some acute angle α (we will general this afterwards).



Then, we can define $\cos\alpha$ to be the x-coordinate of A, and $\sin\alpha$ to be the y-coordinate of A. Notice that these definitions work because if you drop a perpendicular C from A to OB, we have right triangle $\triangle OAC$. This triangle has $\angle OCA = 90$ and $\angle AOC = \alpha$, so we know from the previous definition of cos that $OC = \cos\alpha$, and that $AC = \sin\alpha$.

In order to generalize the trigonometric functions to non-acute values of α, we can let α be the angle OA makes with the positive x-axis (i.e., segment OB). This means that if A is in the second quadrant, then $90 < \alpha < 180$, if A is in the third quadrant, then $180 < \alpha < 270$, and if A is in the fourth quadrant, then $270 < \alpha < 360$. Then, we define $\cos\alpha$ to be the x-coordinate of A, and $\sin\alpha$ to be the y-coordinate of A.



We can even generalize these functions to negative values of α and values of $\alpha > 360$. For negative values, you move A clockwise instead of counterclockwise around the unit circle, and for values $\alpha > 360$, you move A multiple revolutions around the unit circle.

Notice the identity $\sin(\alpha) = \sin(360 + \alpha)$, and similar for the other trigonometric functions. This is because if you move point A a full rotation around the unit circle, or

360 degrees, it is the same point, so it has the same x-coordinate and y-coordinate as the original point A, and hence the same trigonometric values.

Also notice that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{the } y\text{-coordinate of } A}{\text{the } x\text{-coordinate of } A} = \text{the slope of } OA$$

Which gives us the generalization for tan in the unit circle. The generalization of sec, csc, and cot follow naturally since they are the reciprocals of cos, sin, and tan, respectively.

Problem 3.1 (2019 AIME II Problem 10)**.** There is a unique angle θ between $0°$ and $90°$ such that for nonnegative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is $\frac{p}{q}$, where p and q are relatively prime integers. Find $p+q$.

Solution. So first we can find out where $\tan\theta$ is positive. It is positive in the 1st and 3rd quadrants because $\tan\theta = \frac{\sin\theta}{\cos\theta}$. This means that $\tan\theta$ can only be positive when $\sin\theta$ and $\cos\theta$ are either both positive or both negative. This only occurs in the 1st and 3rd quadrants. So now $\tan\theta$ is positive when $0° < \theta < 90°$ and $180° < \theta < 270°$. If we take everything $\mod 180$, we see that $\tan\theta$ is positive at $0° < \theta < 90°$. This means that for all n that are multiples of 3, $\tan 2^n\theta \bmod 180$ are all the same value! This means that we can write $2^0\theta = 2^3\theta \bmod 180 \implies 7\theta = 0 \bmod 180$. We also know that θ has to be in between $0°$ and $90°$. This leaves us $7\theta = 180, 360, 540$. Now doing quick checks by plugging in the three values we have for theta into $\tan 2^n\theta$, we can see that the only value of θ that works is $\frac{540}{7}$ which means that our answer is $540 + 7 = \boxed{547}$ □

3.2 Trigonometric Identities with the Unit Circle Definition

There are multiple trigonometric identities that follow from the unit circle definition. It should be emphasized that this identities must be understood, *not* memorized.

$$\cos\theta = \cos(-\theta)$$

When you negate θ, this is the equivalent of reflecting the point over the x-axis, which does not impact the x-coordinate of the point. Thus, $\cos = \cos(-\theta)$.

$$\sin\theta = -\sin(-\theta)$$

Similar to before, negating θ is the equivalent of reflecting over the x-axis. However, since sin is the y-coordinate, the y-coordinate gets negated as well.

A few other trigonometric identities that can be proven with the Unit Circle definition are $\sin\theta = \sin(180-\theta)$ and $\cos\theta = -\cos(180-\theta)$. You will prove these as an exercise later.

Another simply amazing trigonometric identity is:

$$\sin\theta = \cos(90-\theta)$$

The proof of this is ingenious: simply reflect any point on the unit circle across the line $x = y$. It will land on the circle (this is trivial by coordinates) with its x and y coordinates reversed, but the angle of the point becomes $90-\theta$. Thus, $\sin\theta = \cos(90-\theta)$. Read over this argument a few times, and make sure you truly understand it, as this identity is extremely important.

Finally, the most famous trigonometric identity, also known as the Pythagorean identity, is:

$$\sin^2\theta + \cos^2\theta = 1$$

(Here, $\sin^2\theta = (\sin\theta)^2$ and similar for cos. Get used to this notation, as you will see it everywhere)

This short yet powerful trigonometric identity follows directly from the unit circle definition. Note that the equation of the unit circle (in Cartesian coordinates) is just $x^2 + y^2 = 1$. Thus, $\sin^2\theta + \cos^2\theta = 1$, from the definition of sin and cos. Remember this identity, as it is used almost everywhere in trigonometry.

Two more identities that follow from this identity are $\sec^2\theta - \tan^2\theta = 1$ and $\csc^2\theta - \cot^2\theta = 1$. You will prove these as exercises later on.

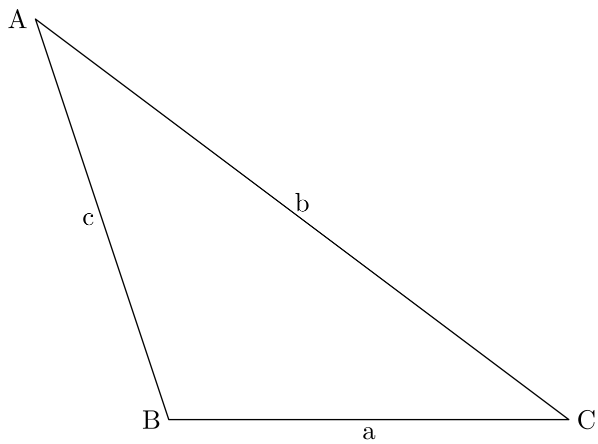
Theorem 3.1

$$-1 \le \sin\theta \le 1$$

$$-1 \le \cos\theta \le 1$$

4 Law of Sines

This is yet another useful formula in trigonometry. It can be used very often and is often useful when one is given 2 angles and 1 corresponding side.



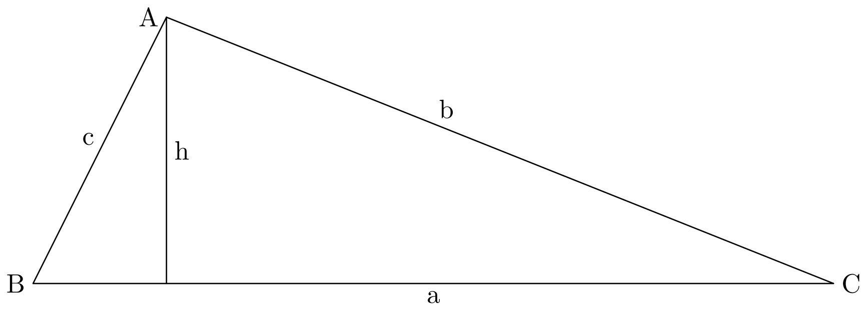
Call BC as a, AC as b, and AB as c. This isn't randomly lettering the sides, but the side lengths are named with the lower letter of the opposite angle.

Theorem 4.1 (Law of Sines)
Given a triangle with side lengths a, b, and c and angles A, B, C, the following statement holds true:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Let's try and prove this!

Proof. Let h be the altitude from Point A to segment a.



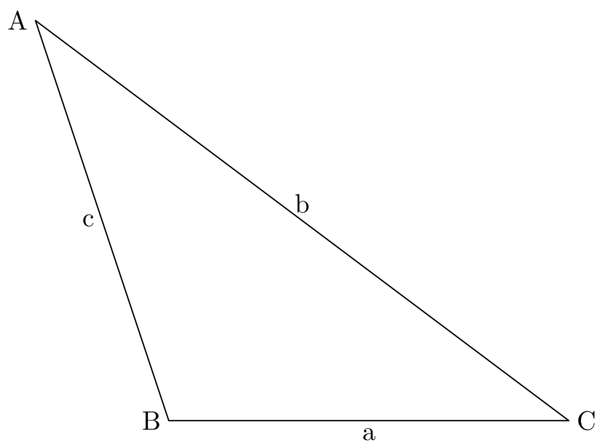
Using the sin function we can write $\sin B = \frac{h}{c}$ and we can also do the same for $\sin C = \frac{h}{b}$. This can be written as $h = c \sin B$ and $h = b \sin C$. Letting the 2 equations equal to one another we find

$$c \sin B = b \sin C \implies \frac{b}{\sin B} = \frac{c}{\sin C}$$

We can do the same thing and replace A and B and find that $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$. This gives us the same 2 equations except that b and a are switched around. From this we know that $\frac{a}{\sin A} = \frac{c}{\sin C}$ would hold true. Now we can piece everything together and find that $\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$ is true! □

5 Law of Cosines

This is a formula which let's us figure out all of the angles and side lengths of a triangle when we are given only 2 side lengths and the included angle's measure! We will be using the same notation as the notation we used in Law of Sines

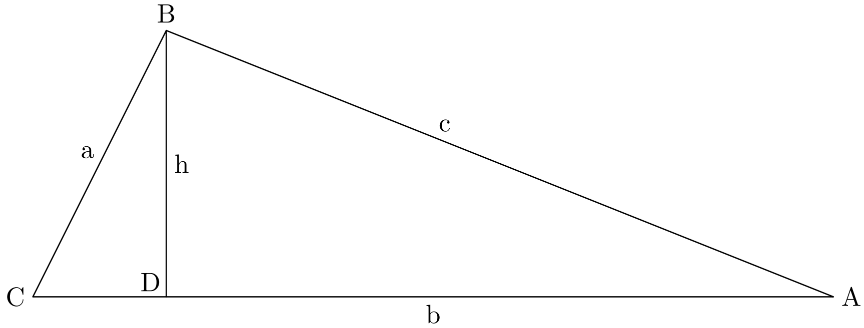


Theorem 5.1 (Law of Cosines)

Given a triangle with side lengths $a, b,$and c, and angles $A, B,$and C, the following statements holds true:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof. Again, let's let h be the altitude from Point B to segment b.



First we can figure out was the value of $\cos C$ is. That is going to be equal to $\frac{CD}{a} \implies CD = a\cos C$. Now we can also see that $b = DA + CD$ so we can rewrite our equation to be $DA = b - a\cos C$. Looking at $\sin C$, we get $\frac{BD}{a}$ or $BD = a\sin C$. Now if we look at $\triangle BAD$, we see that $(AD)^2 + (BD)^2 = c^2$. Notice that we have what DA and BD are, so plugging that into the equation we get:

$$c^2 = (a\sin C)^2 + (b - a\cos C)^2 \implies c^2 = a^2\sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C$$

$$c^2 = a^2\left(\sin^2 C + \cos^2 C\right) + b^2 - 2ab\cos C$$

Part of this looks similar to what we have learned. Notice that we know that $\sin^2\theta + \cos^2\theta = 1$, which means that $c^2 = a^2(1) + b^2 - 2ab\cos C$ and here we have finished the proof. □

It should be noted that the above equation can be rearranged into

$$\cos C = -\frac{c^2 - a^2 - b^2}{2ab}$$

This (often times used in conjunction with the unit circle trigonometric identities described above) proves very useful in many cases, especially in problems that ask to relate side lengths given only angles. If you see a pair of supplementary angles and are asked about side lengths, then consider using this form of the Law of Cosines.

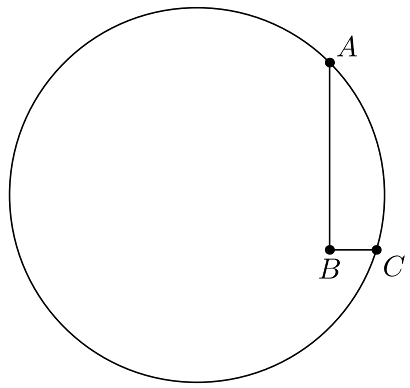
§6 Exercises

Problem 6.1 (Classic)**.** What is the value of $\sin 45°$? What about $\cos 45°$?

Problem 6.2 (Classic)**.** Prove that $\sin\theta = \sin(180 - \theta)$ and $\cos\theta = -\cos(180 - \theta)$

Problem 6.3 (Classic)**.** Prove that $\sec^2\theta - \tan^2\theta = 1$ and $\csc^2\theta - \cot^2\theta = 1$.

Problem 6.4 (1983 AIME)**.** A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.



Problem 6.5 (Stewart's Theorem)**.** Given $\triangle ABC$ and a point D on segment BC, prove that

$$(BD)(CD)(BC) + (AD)^2(BC) = (AC)^2(BD) + (AB)^2(CD)$$

Problem 6.6 (1988 AHSME)**.** If $\sin(x) = 3\cos(x)$ then what is $\sin(x) \cdot \cos(x)$?

Problem 6.7 (1999 AHSME)**.** Let x be a real number such that $\sec x - \tan x = 2$. Then what is $\sec x + \tan x$

Problem 6.8 (1963 AHSME)**.** In $\triangle ABC$, side $a = \sqrt{3}$, side $b = \sqrt{3}$, and side $c > 3$. Let x be the largest number such that the magnitude, in degrees, of the angle opposite side c exceeds x. Then what is x?

Problem 6.9 (2000 AMC 12)**.** A circle centered at O has radius 1 and contains the point A. The segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on OA and BC bisects $\angle ABO$, then what is OC?